Generalisations of Chevalley Restriction Theorem

Fix a reductive Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ of char 0.

Fix Cartan $\mathfrak{h}$, $\exp(\mathfrak{g}) = G$, Weyl group W.

Starting with closed embedding
$$\mathfrak{h} \longrightarrow \mathfrak{g}$$
we can construct a map
$$\mathbb{C}[\mathfrak{g}] \longrightarrow \mathbb{C}[\mathfrak{h}]$$
which is surjective.

Restricting this map to invariant polynomials,
$$\mathbb{C}[\mathfrak{g}]^G \longrightarrow \mathbb{C}[\mathfrak{h}]^{N_G(\mathfrak{h})} = \mathbb{C}[\mathfrak{h}]^W$$

<u>Chevalley restriction theorem</u>:

The map $\phi : \mathbb{C}[\mathfrak{g}]^G \longrightarrow \mathbb{C}[\mathfrak{h}]^W$ is an isomorphism.

OR

There is an isomorphism of schemes
$$\mathfrak{h}/\!/W \longrightarrow \mathfrak{g}/\!/G.$$

<u>Example</u>: $\mathfrak{g} = \mathfrak{gl}_n$, $\mathfrak{h} = $ Diagonal, $W = S_n$

$\mathbb{C}$ - points of $\mathfrak{h}/\!/W$ = W - orbits in $\mathfrak{h}$

= unordered n - tuples of complex numbers

$\mathbb{C}$ - points of $\mathfrak{g}/\!/G$ = Closed G - orbits in $\mathfrak{g}$

= Parametrised by the Jordan normal form s.t. the size of each block is 1.

- $$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & t^{-1} \end{pmatrix} \begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & 1 \\ & & & \lambda \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & t \end{pmatrix} = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & t \\ & & & \lambda \end{pmatrix}$$

  If $t \to 0$, we get $\begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & 0 \\ & & & \lambda \end{pmatrix}$ in the orbit closure.

- Orbits of diagonal matrices are closed because of Cayley - Hamilton theorem and the fact that the min. poly. has no <u>repeated</u> roots.

Proofs of CRT : $(\mathfrak{g} = \mathfrak{gl}_n)$

I) Prove that ϕ is injective and surjective.

Injectivity : Suppose $f \in \mathbb{C}[\mathfrak{g}]^G$ s.t $f|_{\mathfrak{h}} = 0$.

$\Rightarrow f(G \cdot \mathfrak{h}) = 0$

But $G \cdot \mathfrak{h}$ is dense in $\mathfrak{g}$.

$\Rightarrow f = 0$

Surjectivity:

$\mathbb{C}[\mathfrak{h}]^W$ is generated as an algebra by the elementary functions

$$p_k = d_1^k + d_2^k + \dots + d_n^k .$$

But, this is exactly the image of the poly.

$Tr(A^k)$.

II) $\mathfrak{h} /\!/ W \longrightarrow \mathfrak{g} /\!/ G$ is a closed embedding.

It is a bijection on $\mathbb{C}$-valued points.

$\mathfrak{g} /\!/ G$ is reduced.

Hence, it's an isomorphism.

III) Construct an inverse map

$$\mathbb{C}[\mathfrak{h}]^W \longrightarrow \mathbb{C}[\mathfrak{g}]^G .$$

To be <u>done later</u>.

Quantization:

$\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{h} \rightarrow$ universal enveloping algebra.

- It has a PBW filtration

Taking associated graded w.r.t. this filtration,

$$gr(\mathcal{U}\mathfrak{g}) = \operatorname{Sym}\mathfrak{g}, \quad gr(\mathcal{U}\mathfrak{h}) = \operatorname{Sym}\mathfrak{h}$$

CRT:
$$\begin{array}{ccc}
(\operatorname{Sym}\mathfrak{g})^G & \xrightarrow{\sim} & (\operatorname{Sym}\mathfrak{h})^W \\
\| & & \| \\
gr(\mathcal{U}\mathfrak{g})^G & & gr(\mathcal{U}\mathfrak{h})^W \\
\| & & \| \\
gr(\mathcal{U}\mathfrak{g}^G) & & gr(\mathcal{U}\mathfrak{h}^W) \\
\| & & \| \\
gr(Z\mathfrak{g}) & & gr(\operatorname{Sym}\mathfrak{h}^W)
\end{array}$$

<u>Theorem</u>: (Harishchandra) There exists an isomorphism $Z\mathfrak{g} \xrightarrow{\sim} (\operatorname{Sym}\mathfrak{h})^W$.

(I'm being slightly hand-wavy about the twisting.)

Generalisations: What if we consider everything in pairs?

$$\mathfrak{h} \times \mathfrak{h} \longrightarrow \mathfrak{g} \times \mathfrak{g}$$

So, we get a map
$$\phi_2 : \mathbb{C}[\mathfrak{g} \times \mathfrak{g}]^G \longrightarrow \mathbb{C}[\mathfrak{h} \times \mathfrak{h}]^W.$$

Is this an isomorphism?

<u>Example</u>: $\mathfrak{g} = \mathfrak{gl}_n$

$$\mathfrak{h} \times \mathfrak{h} \longrightarrow \mathfrak{g} \times \mathfrak{g}$$
$$\rightsquigarrow (\mathfrak{h} \times \mathfrak{h})/\!/W \longrightarrow (\mathfrak{g} \times \mathfrak{g})/\!/G$$

$\mathbb{C}$-points in LHS = W-orbits in $\mathfrak{h} \times \mathfrak{h}$

= n unordered pairs of complex numbers

$\mathbb{C}$ - points in RHS = WILD

Image only consists of pairs of commuting matrices.

$C_2(\mathfrak{g})$ = subscheme of $\mathfrak{g} \times \mathfrak{g}$ defined by the ideal generated by the matrix entries of $[X, Y]$

Not yet known to a reduced scheme!

Let $C_2(\mathfrak{g})^{\text{red.}}$ by the underlying reduced scheme

Consider $(\mathfrak{h} \times \mathfrak{h})/\!/W \longrightarrow C_2(\mathfrak{g})^{\text{red.}}/\!/G$.

Claim : This is a bijection on $\mathbb{C}$ - points.

- Commuting matrices are simultaneously diagonisable.

Theorem : $\mathbb{C}[\,C_2(\mathfrak{g})^{\text{red.}}]^G \longrightarrow \mathbb{C}[\mathfrak{h} \times \mathfrak{h}]^W$ is an isomorphism.

So, we have

$$\mathbb{C}[\mathfrak{h} \times \mathfrak{h}]^W \xrightarrow{\sim} \mathbb{C}[C_2(\mathfrak{g})^{\text{red.}}]^G \xrightarrow[R]{} \mathbb{C}[C_2(\mathfrak{g})]^G.$$

Theorem: R is an isomorphism when:

a) $\mathfrak{g} = \mathfrak{gl}_n$ Domokos, Vaccarino, Gan-Ginzburg

b) $\mathfrak{g} = \mathfrak{sp}_{2n}$ Chau-Chen, Losev

[GG]: $C_2(\mathfrak{gl}_n)/\!/G$ is reduced.

[L]: $C_2(\mathfrak{sp}_{2n})/\!/G$ is reduced.

Constuction of the spectral data map:

Take $\mathfrak{g} = \mathfrak{gl}_n$.

We want to constuct

$$\mathbb{C}[\mathfrak{h}^d]^W \longrightarrow \mathbb{C}[C_d(\mathfrak{g})]^G = R$$

Consider the polynomials in R: $x_{ij,k}$

$1 \leq i,j \leq n, \quad 1 \leq k \leq d$

Consider the polynomial algebra $\mathbb{C}[t_1, \ldots t_d]$ and the associative algebra $\mathfrak{gl}_n(R)$.

$$\begin{aligned} \theta : \mathbb{C}[t_1, \ldots, t_d] &\longrightarrow \mathfrak{gl}_n(R) \\ t_1^{n_1} \cdots t_d^{n_d} &\longmapsto A_1^{n_1} \cdots A_d^{n_d} \end{aligned}$$

$$A_k = (x_{ij,k})$$

$$\begin{aligned} \phi : \mathbb{C}[t_1, \ldots, t_d] &\longrightarrow R \\ f &\longmapsto \det(\theta(f)) \end{aligned}$$

This is a polynomial map of degree n.

We want to construct

$$\begin{array}{ccc} \mathbb{C}[\mathfrak{h}^d]^W & \longrightarrow & R \\ \| & & \\ \left(\mathbb{C}[t_1, t_2, \dots, t_d]^{\otimes n}\right)^{S_n} & & \end{array}$$

<u>Theorem</u>: (Roby) Let M and N be $\mathbb{C}$-algebras and

$$\phi : M \longrightarrow N$$

be a multiplicative polynomial map of degree n.
Then, $\exists$ a lift $\bar{\phi} : (M^{\otimes n})^{S_n} \longrightarrow N$.

s.t. $\bar{\phi}(m \otimes m \cdots \otimes m) = \phi(m)$.

<u>Proof</u>:

$$\begin{aligned} \phi &\in \mathbb{C}[M]_n \otimes N \\ &\cong \mathrm{Sym}^n(M^*) \otimes N \\ &\cong \mathrm{Hom}(\mathrm{Sym}^n(M^*)^*, N) \\ &\cong \mathrm{Hom}(\mathrm{Sym}^n(M), N) \\ &= \mathrm{Hom}((M^{\otimes n})^{S_n}, N). \end{aligned}$$

$$\mathbb{C}[V] \cong \underline{\mathrm{Sym}\, V^*}$$

Hence, we get a map

$$\bar{\phi} : \mathbb{C}[\mathfrak{h}^d]^W \longrightarrow R.$$

<u>Example</u> : Take $d = 1$.

$$\phi : \mathbb{C}[t] \longrightarrow \mathbb{C}[\mathfrak{g}]^G$$
$$f(t) \longmapsto \det(f(A))$$

$$\rightsquigarrow \quad \overline{\phi} : \mathbb{C}[\mathfrak{h}] \longrightarrow \mathbb{C}[\mathfrak{g}]^G$$
$$\mathbb{C}[\mathfrak{h}] = (\mathbb{C}[t]^{\otimes n})^{S_n}$$

By construction,

$$t \otimes t \otimes \cdots \otimes t \longmapsto \det(A)$$
$$(t-\lambda) \otimes \cdots \otimes (t-\lambda) \longmapsto \det(A - \lambda I)$$

$$\text{Coeff. of } \lambda^i \text{ on the left} = \mathrm{Sym}(t \otimes t \cdots \otimes t \otimes \underbrace{1 \cdots \otimes 1}_{i \text{ times}}) \, (-1)^i$$

$$= (-1)^i e_{n-i}$$

$$\text{Coeff of } \lambda^i \text{ on the right} = i^{th} \text{ coeff. of characteristic poly.}$$

The map $\overline{\phi}$ is called the spectral data map.

<u>Theorem</u> : (Domokos, Vaccarino) $\overline{\phi}$ is the inverse of

the Chevalley restriction map.

- The above construction of the spectral data map is highly specific to $\mathfrak{g} = \mathfrak{gl}_n$. What about other $\mathfrak{g}$?

Construction of the spectral data map for $\mathfrak{sp}_{2n}$:
Let V be a symplectic vector space of dim $2n$.
Then, $\mathfrak{gl}(V) = \mathfrak{sp}(V) \oplus \mathfrak{g}'(V)$ ω.

$$x \in \mathfrak{sp}(V) \iff \omega(v, xw) = -\omega(xv, w)$$
$$x \in \mathfrak{g}'(V) \iff \omega(v, xw) = \omega(xv, w)$$

We want to construct
$$\mathbb{C}[\mathfrak{h}^d]^W \longrightarrow \mathbb{C}[C_d(\mathfrak{g})]^G = R$$

$$\theta : \mathbb{C}[t_1, \dots, t_d] \longrightarrow \mathfrak{gl}_{2n}(R)$$
$$t_1^{n_1} \cdots t_d^{n_d} \longmapsto A_1^{n_1} \cdots A_d^{n_d}$$

$$\phi : \mathbb{C}[t_1, \dots t_d]_{even} \longrightarrow R$$
$$f \longmapsto pf(\theta(f))$$

By Roby, $\bar{\Phi} : \left(\mathbb{C}[t_1, \dots, t_d]_{even}^{\otimes n}\right)^{S_n} \longrightarrow R$

$$\left(\mathbb{C}[t_1,\ldots,t_d]^{\otimes n}\right)^{S_n \ltimes (\mathbb{Z}/2)^n}$$

Key step: Show that the Pfaffian is multiplicative.

<u>Theorem</u>: (Chau - Chin)

$$\mathbb{C}[\mathfrak{h}^d]^W \longrightarrow \mathbb{C}[C_d(\mathfrak{g})]^G.$$

Quantisation: Take $d = 2$.

A natural quantisation of $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}]$ is $\mathcal{D}(\mathfrak{h})$

" " " " $\mathbb{C}[\mathfrak{g} \times \mathfrak{g}]$ " $\mathcal{D}(\mathfrak{g})$

We can try to construct

$$\begin{array}{c} \mathcal{D}(\mathfrak{g})^G \longrightarrow \mathcal{D}(\mathfrak{h}^{reg.})^W \\ \Big\downarrow \text{twisting} \\ HC : \mathcal{D}(\mathfrak{g})^G \longrightarrow \mathcal{D}(\mathfrak{h})^W \end{array}$$

<u>Theorem</u>: (Lavoussein - Stafford)

$$\frac{\mathcal{D}(\mathfrak{g})^G}{I} \xrightarrow{\sim} \mathcal{D}(\mathfrak{h})^W$$

where the ideal $I = \left(\mathrm{ad}\,\mathfrak{g} \cdot \mathcal{D}(\mathfrak{g})\right)^G$

Easy calculation: $\underset{\shortparallel}{gr}(I) \subseteq \mathbb{C}[\mathfrak{g} \times \mathfrak{g}]^G$

exactly the radical ideal defining the commuting scheme.

$$\mathbb{C}[C_2(\mathfrak{g})^{red.}]^G \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}\times\mathfrak{h}]^W.$$

Symmetric pairs

Let $\mathfrak{g}$ be reductive and $\tau : \mathfrak{g} \to \mathfrak{g}$ be an involution.

Then, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$

s.t. $[\mathfrak{k},\mathfrak{k}] \subseteq \mathfrak{k}$,

$[\mathfrak{k},\mathfrak{p}] \subseteq \mathfrak{p}$,

$[\mathfrak{p},\mathfrak{p}] \subseteq \mathfrak{k}$

$(\mathfrak{g},\mathfrak{k})$ is called a symmetric pair.

Examples:

1) Diagonal pairs:
$(\mathfrak{g},\mathfrak{k}) = (\mathfrak{g}\times\mathfrak{g}, \mathfrak{g})$

2) Bilinear forms:
$(\mathfrak{g},\mathfrak{k}) = (\mathfrak{gl}_{2n}, \mathfrak{so}_{2n}), (\mathfrak{gl}_{2n}, \mathfrak{sp}_{2n})$

3) Polarisation:
$(\mathfrak{g},\mathfrak{k}) = (\mathfrak{sp}_{2n}, \mathfrak{gl}_n), (\mathfrak{so}_{2n}, \mathfrak{gl}_n)$

4) Direct sum:
$(\mathfrak{g},\mathfrak{k}) = (\mathfrak{gl}_{n+k}, \mathfrak{gl}_n \times \mathfrak{gl}_k)$, etc

Let $\mathfrak{k} = \mathrm{Lie}(K)$, $\mathfrak{p} = \mathrm{Lie}(P)$.

Then, $K \curvearrowright \mathfrak{p}$.

There exists $\mathfrak{h} \subseteq \mathfrak{p}$
$\hookrightarrow$ maximal subspace of pairwise commuting semisimple elements.

- All such choices of $\mathfrak{h}$ are conjugate under K.

Define $W := N_K(\mathfrak{h}) / C_K(\mathfrak{h})$

Little Weyl group of the pair $(\mathfrak{g}, \mathfrak{k})$.

Given $\mathfrak{h} \hookrightarrow \mathfrak{p}$, we can construct
$\mathbb{C}[\mathfrak{p}]^K \longrightarrow \mathbb{C}[\mathfrak{h}]^W$

<u>Theorem</u>: The map $\mathbb{C}[\mathfrak{p}]^K \longrightarrow \mathbb{C}[\mathfrak{h}]^W$ is an isomorphism.

<u>Theorem</u>: (Pattanayak, Nadimpalli)
$\mathbb{C}[C_d(\mathfrak{p})]^K \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}^d]^W$ is an isomorphism for all the classical pairs above except $(so_n \times so_n, so_n)$ and (so_{2n}, gl_n).

Another direction of generalisation:

Luna - Richardson Theorem

Let G - reductive act on an affine, normal, irreducible variety X. Let $a \in X /\!/ G$ be a principal point and let $x \in \pi^{-1}(a)$ $\qquad \pi : X \longrightarrow X/\!/G$

s.t. orbit of x is closed.

Define $W := N_G(G_x)/G_x$, where G_x is the centraliser of x.

Then,

Theorem:
$$\mathbb{C}[X]^G \xrightarrow{\sim} \mathbb{C}[X^{G_x}]^H .$$

Principal point: $a \in X/\!/G$ is called principal if there is a neighbourhood $a \in U \subseteq X/\!/G$ s.t. for all $b \in U$, the closed orbit points above a and b have conjugate centralisers.