Chevalley restriction theorem for algebraic varieties and Cherednik algebras

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The classical Chevalley restriction theorem

Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{C}$.
Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

Let G be a connected algebraic group such that $\mathrm{Lie}(G) = \mathfrak{g}$.
Let W be the Weyl group associated with the above data.

We have a natural restriction map:

$$\mathbb{C}[\mathfrak{g}] \longrightarrow \mathbb{C}[\mathfrak{h}],$$

which induces a map between invariant rings:

$$\mathrm{res} : \mathbb{C}[\mathfrak{g}]^G \longrightarrow \mathbb{C}[\mathfrak{h}]^W.$$

<u>Theorem</u>: (Chevalley) The map 'res' is an isomorphism.

Geometrically, this says that we have an isomorphism of schemes: $\mathfrak{h}/\!/W \xrightarrow{\sim} \mathfrak{g}/\!/G$.

Example : Consider $\mathfrak{g} = \mathfrak{gl}_n$ $\mathfrak{h}$ = diagonal matrices

$G = GL_n(\mathbb{C})$ $W = S_n$

In this case, the above theorem is equivalent to the fact that conjugation invariant polynomial functions on the space of $n \times n$ matrices are generated by traces of powers.

In the general case, the theorem is proved by

- Highest weight representation theory
- Geometric properties of the Springer resolution
- Construction of an explicit inverse.

Construction of the representation scheme

Let $X = \mathrm{Spec}(R)$ be an affine algebraic variety over $\mathbb{C}$. Let V be an n-dimensional vector space over $\mathbb{C}$.

Def^n: (Finkelberg, Ginzburg)
The representation scheme rep_X of X over V is defined as the affine scheme that parametrizes algebra homomorphisms

$$\phi : R \longrightarrow \mathfrak{gl}(V).$$

Geometrically, rep_X is the scheme that parametrizes the data of a finite length $\mathcal{O}_X$-sheaf $\mathcal{F}$ and a vector space isomorphism

$$V \xrightarrow{\sim} \Gamma(X, \mathcal{F}).$$

Examples:

1) $X = \mathbb{A}^1$ $\qquad R = \mathbb{C}[t]$

$$\mathrm{rep}_X = \mathfrak{gl}(V)$$

2) $X = \mathbb{C}^*$ $\qquad R = \mathbb{C}[t, t^{-1}]$

$$\mathrm{rep}_X = GL(V)$$

3) $X = \mathbb{A}^k$ $\qquad R = \mathbb{C}[t_1, t_2, \ldots, t_k]$

$\mathrm{rep}_X =$ commuting scheme of k-tuples of elements of $\mathfrak{gl}(V)$.

$$\text{``} = \{ (x_1, \ldots, x_k) \in (\mathfrak{gl}(V))^k : [x_i, x_j] = 0 \;\; \forall\, i, j \} \text{''}$$

Consider the symmetric power

$$X^{(n)} = X^n /\!/ S_n .$$

Having fixed an identification $\mathfrak{gl}(V) \simeq \mathfrak{gl}_n$, we can construct a closed imbedding:

$$\mathrm{Res} : X^n \longrightarrow \mathrm{rep}_X$$

$$x = (x_1, x_2, \dots, x_n) \longmapsto \phi_x : R \longrightarrow \mathfrak{gl}(V)$$

$$r \longmapsto \begin{bmatrix} r(x_1) & & & \\ & r(x_2) & & \\ & & \ddots & \\ & & & r(x_n) \end{bmatrix}$$

This map gives rise to a map on quotients:

$$\overline{\mathrm{Res}} : X^{(n)} \longrightarrow \mathrm{rep}_X /\!/ GL(V).$$

While the map Res depends on the choice of the identification $\mathfrak{gl}_n \simeq \mathfrak{gl}(V)$, the map $\overline{\mathrm{Res}}$ is completely canonical.

When $X = \mathbb{A}^1$, $\overline{\mathrm{Res}}$ is exactly the Chevalley restriction map res for $\mathfrak{g} = \mathfrak{gl}_n$.

Theorem: The map $\overline{\text{Res}}$ is an isomorphism of schemes.

The proof involves the constuction of an explicit inverse.
For $X = \mathbb{A}^1$, this makes use of Deligne's spectral data map, that essentially takes a representation to its support cycle.

Algebraically, this inverse was constructed independently by Domokos and Vaccarino, via polarization of the determinant polynomial.

Corollary: The quotient $\text{rep}_X /\!/ GL(V)$ is a reduced scheme.

Scheme of symplectic representations

Let $X = \mathrm{Spec}(R)$ be an affine algebraic variety with a $\mathbb{Z}/2$ - action.

This gives an eigenspace decomposition :

$$R = R_+ \oplus R_- .$$

Let V be a symplectic vector space of dimension $2n$, with symplectic form ω.

Let $\mathfrak{g} = \mathfrak{sp}(V)$ = symplectic Lie algebra

$$= \{ x \in \mathfrak{gl}(V) : \text{For all } v, w \in V, \ \omega(xv, w) = -\omega(v, xw) \}$$

$$\mathfrak{g}^{+} = \{ x \in \mathfrak{gl}(V) : \text{For all } v, w \in V, \ \omega(xv, w) = \omega(v, xw) \}$$

Then, $\mathfrak{gl}(V) = \mathfrak{g} \oplus \mathfrak{g}^{+}$

Def^n: The symplectic representation scheme srep_X of X over V is defined as the affine scheme parametrizing algebra homomorphisms

$$\phi : R \longrightarrow \mathfrak{gl}(V)$$

such that $\phi(R^{+}) \subseteq \mathfrak{g}^{+}$ and $\phi(R^{-}) \subseteq \mathfrak{g}$.

Examples :

1) $X = \mathbb{A}^1$ with $\mathbb{Z}/2$ acting by sign
$\mathrm{srep}_X = \mathfrak{sp}(V)$

2) $X = \mathbb{A}^1$ with $\mathbb{Z}/2$ acting trivially
$\mathrm{srep}_X = \mathfrak{g}^+$

3) $X = \mathbb{C}^*$ with $\mathbb{Z}/2$ acting by inverse
$\mathrm{srep}_X = Sp(V)$, the symplectic group

4) $X = \mathbb{C}^*$ with $\mathbb{Z}/2$ acting trivially
$\mathrm{srep}_X = (\mathfrak{g}^+)^*$

The variety X has a $\mathbb{Z}/2$-action. Hence, on X^n, we get an action of the semi-direct product

$$W := (\mathbb{Z}/2)^n \rtimes S_n,$$

which is exactly the Weyl group of Type C.

As before, we have a natural map

$$\mathrm{Res}: \quad X^n /\!/ W \longrightarrow \mathrm{Srep}_X /\!/ Sp(V).$$

When $X = \mathbb{A}^1$ with $\mathbb{Z}/2$ acting by sign, this is exactly the Chevalley restriction map for $\mathfrak{g} = \mathfrak{sp}_{2n}$.

Theorem: [G] The map Res is an isomorphism of schemes.

Corollary: The scheme $\mathfrak{sp}_X /\!/ Sp(V)$ is reduced.

- The construction of the inverse map relies on a certain spectral data map constructed by Ngô and Chen.

This map essentially involves the polarization of a certain Pfaffian norm.

Why care about these representation schemes?

Let X be a smooth algebraic curve.

Etingof defined a sheaf of associative algebras on X^n, known as global Cherednik algebras, that are deformations of the algebra $D(X^n) \rtimes S_n$.

Finkelberg and Ginzburg showed that these Cherednik algebras can be constructed via Hamiltonian reduction from $D(\mathrm{rep}_X)$.

They used this to construct a category of D-modules that generalize Lusztig's character sheaves.

When X has a $\mathbb{Z}/2$-action, similar Cherednik algebras were defined by Etingof as deformations of $\mathcal{D}(X^n) \rtimes W$.

Then, these can be obtained via Hamiltonian reduction of $\mathcal{D}(\mathrm{rep}_X)$.

In either case, if $X = \mathbb{A}^1$ (resp. $\mathbb{C}^\times$), the corresponding results about rational (resp. trigonometric) Cherednik algebras are recovered.

Thank You!