Hall algebra of $\mathfrak{sl}_2$-modules over positive characteristic and shifted quantum loop algebras

Pallav Goyal

Department of Mathematics
University of California Riverside

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Main result

- Ringel, Green (1995)

$$\mathcal{U}_v(\mathfrak{g}_Q^+) \hookrightarrow \mathrm{Hall}(Q - mod)$$

- Bridgeland (2013)

$$\mathcal{U}_v(\mathfrak{g}_Q) \hookrightarrow \mathrm{Hall}(\mathcal{P}_2(Q - mod))$$

- G.-Samuelson (2025)

$$\mathcal{U}_v(\hat{\mathfrak{sl}}_2)_{1,1} \hookrightarrow \mathrm{Hall}(Q_{Rud}/R - mod) \hookrightarrow \mathrm{Hall}(\mathfrak{sl}_2\text{-}mod_{rest})$$

Hall algebras

$\mathcal{C}$ - Small abelian finitary category (All Hom and Ext groups are finite)
$\mathcal{H}_{\mathcal{C}}$ - $\mathbb{C}$-vector space with basis given by $[M]$, where $M \in \mathrm{Ob}(\mathcal{C})/\sim$
Define an algebra structure on $\mathcal{H}_{\mathcal{C}}$ as follows:

$$[M] \cdot [N] := \sum_R P^R_{M,N}[R],$$

where,

$$P^R_{M,N} := |\{L \subseteq R : L \simeq N, R/L \simeq M\}|,$$

and extended linearly on $\mathcal{H}_{\mathcal{C}}$.

Easy exercise

This product is associative.

The $\mathbb{C}$-algebra $\mathcal{H}_{\mathcal{C}}$ is called the Hall algebra of the category $\mathcal{C}$.

$\mathbb{F}_q$ - Finite field of size $q = p^n$ (odd)
Q - Quiver without oriented cylces
$\mathcal{C}$ - Category of f.d. representations of Q over $\mathbb{F}_q$.
$\mathcal{H}_Q$ - Hall algebra of $\mathcal{C}$

Example 1: $Q = A_1$

This is the quiver with a single vertex with no arrows.

$$\mathcal{C} \simeq Vect \implies \mathcal{H}_Q \simeq \mathbb{C}[t]$$

Example 2: $Q = A_2$

$$1 \longrightarrow 2$$

Q has two simple representations: S_1 and S_2, each suppoerted at the respective vertex.

$$[S_2] \cdot [S_1] = [S_1 \oplus S_2]$$
$$[S_1] \cdot [S_2] = [S_1 \oplus S_2] + [P],$$

where P is the indecomposable representation:

$$\mathbb{F}_q \xrightarrow{\text{Id}} \mathbb{F}_q \ .$$

An easy computation shows that:

$$[S_1]^2 \cdot [S_2] - (q+1)[S_1] \cdot [S_2] \cdot [S_1] + q[S_2] \cdot [S_1]^2 = 0,$$

$$[S_2]^2 \cdot [S_1] - (q+1)[S_2] \cdot [S_1] \cdot [S_2] + q[S_1] \cdot [S_2]^2 = 0.$$

A_Q - Adjacency matrix of Q
$C_Q := 2I - A_Q$ - Euler matrix of Q

This matrix C_Q is a generalized Cartan matrix, and so, we can associate with it a (possibly infinite-dimensional) Kac-Moody Lie algebra $\mathfrak{g}_Q$.

Its universal enveloping algebra $\mathcal{U}(\mathfrak{g}_Q)$ has a canonical quantization as a Hopf algebra, denoted by $\mathcal{U}_v(\mathfrak{g}_Q)$, and called the quantum Kac-Moody Lie algebra.

Explicitly, we can describe $\mathcal{U}_v(\mathfrak{g}_Q)$ as the $\mathbb{C}(v)$-algebra generated by $\{E_i, F_i, K_i^{\pm 1}\}_{i \in vertex(Q)}$ satisfy the relations:

$$K_i K_j = K_j K_i \tag{1}$$

$$K_i E_j K_i^{-1} = v^{c_{ij}} E_j \tag{2}$$

$$K_i F_j K_i^{-1} = v^{-c_{ij}} F_j \tag{3}$$

$$[E_i, F_j] = \delta_{i,j} \frac{K_i - K_i^{-1}}{v - v^{-1}} \tag{4}$$

$$\sum_{l=0}^{1-c_{ij}} \binom{1-c_{ij}}{l}_v E_i^l E_j E_i^{1-c_{ij}-l} = 0 \tag{5}$$

$$\sum_{l=0}^{1-c_{ij}} \binom{1-c_{ij}}{l}_v F_i^l F_j F_i^{1-c_{ij}-l} = 0. \tag{6}$$

The subalgebra generated by the E_i's is denoted by $\mathcal{U}_v(\mathfrak{g}_Q^+)$ and is called the positive part of $\mathcal{U}_v(\mathfrak{g}_Q)$.

Suppose the product on the Hall algebra $\mathcal{H}_Q$ is twisted as follows:

$$[M] * [N] := q^{\langle M,N\rangle/2}[M] \cdot [N],$$

where the additive Euler form $\langle \cdot, \cdot \rangle$ is defined via:

$$\langle M, N\rangle = \dim(\mathrm{Hom}(M, N)) - \dim(\mathrm{Ext}^1(M, N)).$$

Theorem (Ringel, Green)

Specializing at $v = q^{1/2}$, there exists an injective (Hopf) algebra homomorphism:

$$\Phi_Q : \mathcal{U}_v(\mathfrak{g}_Q^+) \to \mathcal{H}_Q,$$

where we use the twisted product on the Hall algebra. Furthermore, the above map is an isomorphism exactly when Q is an ADE quiver (with arbitrary orientation).

A limitation of the previous construction is that it only produces ‘half’ of the quantum group. Algebraically, the whole quantum group can be obtained from either of its ‘halves’ via a Drinfeld double construction. Morally, this Drinfeld double should be some invariant of a 2-periodic version of the derived category $D^b(\mathcal{C})$.

An approach by Bridgeland: Let $\mathcal{P}_2(\mathcal{C})$ be the category of 2-periodic complexes of projective objects in $\mathcal{C}$. Define the algebra:

$$\mathcal{DH}(\mathcal{C}) := \frac{\mathcal{H}_{\mathcal{P}_2(\mathcal{C})}[[M_\bullet]^{-1} : H_*(M_\bullet) = 0]}{([M_\bullet] - 1 : H_*(M_\bullet) = 0, M_\bullet \simeq M_\bullet[1])}.$$

Theorem (Bridgeland)

Specializing at $v = q^{1/2}$, there exists an injective algebra homomorphism:

$$\Phi_Q : \mathcal{U}_v(\mathfrak{g}_Q) \to \mathcal{DH}(\mathcal{C}).$$

Furthermore, the above map is an isomorphism exactly when Q is an ADE quiver (with arbitrary orientation).

Representation theory of $\mathfrak{sl}_2$

$\mathcal{C}$ - Category of restricted representations of $\mathfrak{sl}_2 := \mathfrak{sl}_2(\mathbb{F}_q)$
Explicitly, if $\mathfrak{sl}_2 = \mathbb{F}_q\langle e, f, h\rangle$, the category $\mathcal{C}$ consists of those those $\mathfrak{sl}_2$-representations which are annihilated by the elements:

$$e^p, f^p, h^p - h$$

in the universal enveloping algebra $\mathcal{U}\mathfrak{sl}_2$. It is well known that the category $\mathcal{C}$ is not semisimple.

Weyl Modules

$V_n := \mathbb{F}_q[x, y]_n$ for some $n \geq 0$
We have a natural $\mathfrak{sl}_2$-action on V_n:

$$e \mapsto y\partial_x, f \mapsto x\partial_y, h \mapsto y\partial_y - x\partial_x.$$

The spaces V_n are known as Weyl modules.

Theorem (Jacobson)

1. *V_n is an indecomposable representation for all $n \in \mathbb{Z}_{\geq 0}$.*
2. *V_n is a simple representation if and only if $n < p$.*
3. *The set $\{V_0, V_1, \ldots, V_{p-1}\}$ is a complete list of simple objects in the category $\mathcal{C}$.*

Theorem (Pollack)

1. *V_{p-1} is both projective and injective in $\mathcal{C}$. In particular, for all i,*

$$\mathrm{Ext}^1(V_i, V_{p-1}) = \mathrm{Ext}^1(V_{p-1}, V_i) = 0.$$

2. *For $0 \leq i, j \leq p-2$,*

$$\dim(\mathrm{Ext}^1(V_i, V_j)) = \begin{cases} 2 & \text{if } i+j = p-2 \\ 0 & \text{otherwise} \end{cases}.$$

Example

Fix i, j such that $i + j = p - 2$. Consider the representation:

$$V_{p+i} = \mathbb{F}_q \langle x^{p+i}, x^{p+i-1}y, \cdots, x^p y^i, x^{p-1}y^{i+1}, \cdots, x^{i+1}y^{p-1}, x^i y^p, \cdots, xy^{p+i-1}, y^{p+i} \rangle$$

Example

Fix i, j such that $i + j = p - 2$. Consider the representation:

$$V_{p+i} = \mathbb{F}_q\langle x^{p+i}, x^{p+i-1}y, \cdots, x^p y^i, x^{p-1}y^{i+1}, \cdots, x^{i+1}y^{p-1}, x^i y^p, \cdots, xy^{p+i-1}, y^{p+i}\rangle$$

Both of the sets of highlighted vectors span subrepresentations of V_{p+i} that are isomorphic to V_i. Furthermore, the quotient of V_{p+i} by these subrepresentations is isomorphic to V_j.

Pollack's theorem implies that we can express the category $\mathcal{C}$ as a direct sum:

$$\mathcal{C} = \tilde{\mathcal{C}} \oplus \bigoplus_{i=0}^{(p-3)/2} \mathcal{C}_i,$$

where,

$\mathcal{C}_i$ - The subcategory of $\mathcal{C}$ generated by V_i and V_{p-2-i} under extensions

$\tilde{\mathcal{C}}$ - The subcategory generated by V_{p-1}

As a result, the Hall algebras of these categories are related via the equality:

$$\mathcal{H}_{\mathcal{C}} = \mathcal{H}_{\tilde{\mathcal{C}}} \otimes \bigotimes_{i=0}^{(p-3)/2} \mathcal{H}_{\mathcal{C}_i}.$$

As the category $\tilde{\mathcal{C}}$ is equivalent to *Vect*, the algebra $\mathcal{H}_{\tilde{\mathcal{C}}}$ is isomorphic to a polynomial algebra in one variable.

Theorem (Rudakov)

For all $0 \leq i \leq (p-3)/2$, *the category* $\mathcal{C}_i$ *is equivalent to the category* $\mathrm{Rep}(Q_{Rud}/R)$, *where:*

$$Q_{Rud} = \bullet \begin{array}{c} \xrightarrow{\;e\;} \\ \xrightarrow{\;f\;} \\ \xleftarrow{\;e'\;} \\ \xleftarrow{\;f'\;} \end{array} \bullet ,$$

and the set R *of relations is given by:*

$$ee' = ff' = 0, e'e = f'f = 0$$

$$ef' = fe', e'f = f'e.$$

The existence of such a quiver and relations follows from the fact that every finite dimensional algebra is Morita equivalent to the path algebra of its Ext-quiver modulo some quadractic relations.

Example

Suppose i and j are such that $i + j = p - 2$.
V - An extension of V_i^m by V_j^n
By Pollack, we know:

$$\mathrm{Ext}^1(V_i^n, V_j^m) \simeq \mathrm{Ext}^1(V_i, V_j)^{mn} \simeq (\mathbb{F}_q^2)^{mn} \simeq (\mathbb{F}_q^{mn})^2.$$

This gives a representation of the above quiver Q_{Rud} having dimension (m, n): The maps e and f can be defined by using these matrices, whereas the maps e' and f' are taken to be zero.

Hall algebra of the Rudakov quiver modulo relations

Goal

Understand the Hall algebra $\mathcal{H}$ of the category $\mathrm{Rep}(Q_{Rud/R})$.

$$Q_{Rud} = \alpha \underset{f'}{\overset{e}{\underset{e'}{\overset{f}{\rightrightarrows}}}} \beta$$

$$ee' = ff' = 0, e'e = f'f = 0$$

$$ef' = fe', e'f = f'e.$$

One of the first steps towards understanding this Hall algebra is to classify the indecomposable objects in $\mathrm{Rep}(Q_{Rud}/R)$.

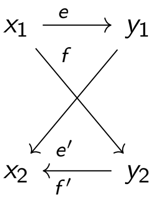
Theorem (Pollack, Rudakov)

All the indecomposable objects in $\mathrm{Rep}(Q_{Rud}/R)$ *can be classified as:*

- *Representations where* $e' = f' = 0$
- *Representations where* $e = f = 0$
- *Two exceptions* M *and* M'*(described below) that are both injective and projective in* $\mathrm{Rep}(Q_{rud}/R)$

Let $V = \mathbb{F}_q\langle x_1, x_2\rangle$ and $W = \mathbb{F}_q\langle y_1, y_2\rangle$ be two-dimensional vector spaces.

$M :=$



The representation M' is obtained from M by flipping along the vertical axis.

The above classification suggests that the algebra $\mathcal{H}$ should be close to being the doubled version of the Hall algebra of the Kronecker quiver:

$$Kron = \alpha \overset{e}{\underset{f}{\rightrightarrows}} \beta .$$

Ringel's theorem implies:

$$\Phi_{Kron} : \mathcal{U}_v(\mathfrak{g}_Q^+) \to \mathcal{H}_{Kron},$$

where $\mathfrak{g}_Q$ is the affine Lie algebra $\hat{\mathfrak{sl}}_2$.

Guess/Hope

The Hall algebra $\mathcal{H}$ should be related to the whole quantum affine algebra $\mathcal{U}_v(\hat{\mathfrak{sl}}_2)$.

Quantum loop algebras

Recall that the quantum affine algebra $\mathcal{U}_v(\hat{\mathfrak{sl}}_2)$ is generated by eight elements $\{E_i, F_i, K_i^{\pm 1}\}_{i=0,1}$, satisfying certain relations.

For $\mathfrak{g}$ simple (for example $\mathfrak{g} = \mathfrak{sl}_2$), the affine Lie algebra $\hat{\mathfrak{g}}$ can be realised as the universal central extension of the loop algebra $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. This universal extension has a different presentation given by Garland, that involves infinitely many generators and relations.

The Garland presentation can be deformed to get a new presentation for $\mathcal{U}_v(\hat{\mathfrak{sl}}_2)$.

Theorem (Beck, Drinfeld)

The quantum affine algebra $\mathcal{U}_v(\hat{\mathfrak{sl}}_2)$ is isomorphic to the quantum loop algebra $\mathcal{U}_v(\mathcal{L}\mathfrak{sl}_2)$, which is the $\mathbb{C}(v)$-algebra generated by elements $E_l, F_l, H_n, K^{\pm 1}, C^{\pm 1/2}$ for $l \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\}$ satisfying the following relations:

$$C^{1/2} \text{ is central}$$

$$KE_kK^{-1} = v^2 E_k, KF_kK^{-1} = v^{-2}F_k, KH_nK^{-1} = H_n$$

$$E_{k+1}E_l - v^2 E_l E_{k+1} = v^2 E_k E_{l+1} - E_{l+1}E_k$$

$$F_{k+1}F_l - v^{-2} F_l F_{k+1} = v^{-2} F_k F_{l+1} - F_{l+1}F_k$$

$$[H_l, E_k] = \frac{[2l]}{l} C^{-|l|/2} E_{k+l}$$

$$[H_l, F_k] = \frac{-[2l]}{l} C^{|l|/2} F_{k+l}$$

$$[H_l, H_k] = \delta_{l,-k} \frac{[2l]}{l} \frac{C^l - C^{-l}}{v - v^{-1}}$$

Theorem (Contd.)

$$[E_k, F_l] = \frac{C^{(k-l)/2}\Psi_{k+l} - C^{(l-k)/2}\Phi_{k+l}}{v - v^{-1}}$$

where the elements Ψ_k and Φ_k are defined via the following generating series:

$$\sum_{k\geq 0} \Psi_k u^k = K \exp\left((v - v^{-1})\sum_{k=1}^{\infty} H_k u^k\right)$$

$$\sum_{k\geq 0} \Phi_{-k} u^k = K^{-1} \exp\left(-(v - v^{-1})\sum_{k=1}^{\infty} H_{-k} u^k\right).$$

Remark

By the work of Burban and Schiffmann, the above isomorphism can be interpreted geometrically in terms of the derived equivalence between the category of representations of the Kronecker quiver and category of coherent sheaves on $\mathbb{P}^1$.

Shifted quantum loop algebra

'Shifted' versions of quantum groups (specifically, truncated quantum Yangians) appeared in the works of Braverman, Finkelberg and Nakajima as a tool to describe quantized Coulomb branches of 3-D $\mathcal{N} = 4$ supersymmetry quiver gauge theories, in terms of generators and relations.

Definition (Finkelberg, Tsymbaliuk)

For any integers b_1, b_2, there exists a shifted quantum loop algebra $\mathcal{U}_v(\mathcal{L}\mathfrak{sl}_2)_{b_1,b_2}$, which has exactly the same generators and relations as $\mathcal{U}_v(\mathcal{L}\mathfrak{sl}_2)$, except we define:

$$u^{b_1} \sum_{k \geq -b_1} \Psi_k u^k = K \exp\left((v - v^{-1}) \sum_{k=1}^{\infty} H_k u^k \right)$$

$$u^{b_2} \sum_{k \geq -b_2} \Phi_{-k} u^k = K^{-1} \exp\left(-(v - v^{-1}) \sum_{k=1}^{\infty} H_{-k} u^k \right).$$

The shifted quantum loop algebra $\mathcal{U}_v(\mathcal{L}\mathfrak{sl}_2)_{1,1}$ turns out to be the 'correct' doubled version of the algebra $\mathcal{U}_v(\hat{\mathfrak{sl}_2}^+)$ that is relevant for describing the Hall algebra $\mathcal{H}$ of $\mathrm{Rep}(Q_{Rud}/R)$.

Define the twisted Hall algebra $\mathcal{H}_{tw}$ which has the multiplication:

$$[V] * [W] := q^{-\det(\dim(V),\dim(W))/2}[V] \cdot [W],$$

for any $V, W \in \mathrm{Ob}(\mathrm{Rep}(Q_{Rud}/R))$.

Theorem (G.-Samuelson)

There exists an injective algebra homomorphism:

$$\Phi : \mathcal{U}_v(\mathcal{L}\mathfrak{sl}_2)_{1,1} \to \mathcal{H}_{tw},$$

when v is specialized to $q^{1/2}$.

Idea of Proof

We describe what the homomorphism Φ maps the generators of $\mathcal{U}_v(\mathcal{L}\mathfrak{sl}_2)_{1,1}$ to.

$$C^{1/2} \mapsto [M']^{1/4} * [M]^{-1/4}$$

$$K \mapsto -([M'] * [M])^{-1/4}$$

For the generators E_n, F_n and H_n, map them to representations where:

- $e' = f' = 0$ if $n \geq 0$
- $e = f = 0$ if $n \leq 0$.

Theorem

Indecomposable representations of the Kronecker quiver over a field F are of three types:

- *Pre-projective representations:*

$$P_n := \; F^n \underset{f}{\overset{e}{\rightrightarrows}} F^{n+1} \; ,$$

where e is the inclusion into the first n coordinates of F^{n+1}, f is the inclusion into the last n.

- *Pre-injective representations:*

$$I_n := \; F^{n+1} \underset{f}{\overset{e}{\rightrightarrows}} F^{n} \; ,$$

where e is the projection onto the first n coordinates of F^{n+1}, f is the projection onto the last n.

Theorem (Contd.)

- *Regular representations:*

$$F^n \overset{e}{\underset{f}{\rightrightarrows}} F^n \ ,$$

where one of e or f is the identity map, and the other is given by an indecomposable $n \times n$ matrix over F.

Remark

Translating back to the category of restricted representations of $\mathfrak{sl}_2$, the representations I_n and P_n exactly correspond to the Weyl modules.

When $n \geq 0$, the homomorphism Φ 'roughly' maps:

- E_n to $[I_n]$
- F_n to $[P_n]$
- H_n to a weighted average of regular representations having dimension (n, n).

The inclusion of the Kronecker quiver into the Rudakov quiver induces a functor:

$$F : \mathrm{Rep}(Kron) \to \mathrm{Rep}(Q_{Rud}/R),$$

which is given by extension by zero on e' and f'. This functor is fully faithful and exact and induces a map between the Ext groups for any $A, B \in \mathrm{Ob}(\mathrm{Rep}(Kron))$:

$$\overline{F} : \mathrm{Ext}^1(A, B) \to \mathrm{Ext}^1(F(A), F(B)).$$

While the functor F does not induce a morphism of Hall algebras, we do have the following:

Proposition

Then, the map $\overline{F}$ is an isomorphism, unless A is a pre-projective representation and B is a pre-injective representation. Furthermore, if the map is not an isomorphism, then either A or B must be a simple representation.

Consequences

- There exists a functor:

$$G : \mathrm{Rep}(Q_{Rud}) \to \mathrm{Rep}(Q_{Rud}/R),$$

which is given by killing those representations where the relations in R do not hold. The functor G induces a morphism on the level of Hall algebras.

By Ringel's theorem, we know that $\mathcal{H}_{Q_{Rud}}$ contatins the positive part of a certain quantum Kac-Moody Lie algebra with Cartan matrix:

$$\begin{pmatrix} 2 & -4 \\ -4 & 2 \end{pmatrix}.$$

Then, the above map allows us to transport relations from this Kac-Moody algebra to the algebra $\mathcal{U}_v(\mathcal{L}\mathfrak{sl}_2)_{1,1}$. In particular, we get that the 5-th v-commutator of E_0 with F_0 vanishes, and vice-versa.

Consequences

- On the new presentation of the quantum loop algebra $\mathcal{U}_v(\mathcal{L}\mathfrak{sl}_2)$, Drinfeld defined a topological co-product which makes it a topological bialgebra. This co-product can be naturally extended to shifted versions of the quantum loop algebra. Our theorem allows us to view this co-product on $\mathcal{U}_v(\mathcal{L}\mathfrak{sl}_2)_{1,1}$ as a version of Green's co-product on the Hall algebra.

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Thank You!