Hall algebras and shifted quantum affine algebras

Pallav Goyal

Department of Mathematics
University of California Riverside

(Joint work with Peter Samuelson)
arXiv:math.RT/2508.09405

Hall algebras

$\mathcal{C}$: Small abelian finitary category (All Hom and Ext groups are finite)
$\mathcal{H}_{\mathcal{C}}$: $\mathbb{C}$-vector space with basis given by $[M]$, where $M \in \mathrm{Ob}(\mathcal{C})/\sim$
Define an algebra structure on $\mathcal{H}_{\mathcal{C}}$ as follows:

$$[M] \cdot [N] := \sum_{R} P_{M,N}^{R} [R],$$

where,

$$P_{M,N}^{R} := |\{L \subseteq R : L \simeq N, R/L \simeq M\}|,$$

and extended linearly on $\mathcal{H}_{\mathcal{C}}$.

Easy exercise

This product is associative.

The $\mathbb{C}$-algebra $\mathcal{H}_{\mathcal{C}}$ is called the Hall algebra of the category $\mathcal{C}$.

$\mathbb{F}_q$: Finite field of size $q = p^n$
Q : Quiver without oriented cylces
$\mathcal{C}$: Category of f.d. representations of Q over $\mathbb{F}_q$.
$\mathcal{H}_Q$: Hall algebra of $\mathcal{C}$

Example 1: $Q = A_1$

This is the quiver with a single vertex with no arrows.

$$\mathcal{C} \simeq Vect \implies \mathcal{H}_Q \simeq \mathbb{C}[t]$$

Example 2: $Q = A_2$

$$1 \longrightarrow 2$$

Q has two simple representations: S_1 and S_2, each supported at the respective vertex.

$$[S_2] \cdot [S_1] = [S_1 \oplus S_2]$$
$$[S_1] \cdot [S_2] = [S_1 \oplus S_2] + [P],$$

where P is the indecomposable representation:

$$\mathbb{F}_q \xrightarrow{\text{Id}} \mathbb{F}_q \ .$$

An easy computation shows that:

$$[S_1]^2 \cdot [S_2] - (q+1)[S_1] \cdot [S_2] \cdot [S_1] + q[S_2] \cdot [S_1]^2 = 0,$$

$$[S_2]^2 \cdot [S_1] - (q+1)[S_2] \cdot [S_1] \cdot [S_2] + q[S_1] \cdot [S_2]^2 = 0.$$

A_Q : Adjacency matrix of Q
$C_Q (:= 2I - A_Q)$: Euler matrix of Q

This matrix C_Q is a generalized Cartan matrix, and so, we can associate with it a (possibly infinite-dimensional) Kac-Moody Lie algebra $\mathfrak{g}_Q$.

Its universal enveloping algebra $\mathcal{U}(\mathfrak{g}_Q)$ has a canonical quantization as a Hopf algebra, denoted by $\mathcal{U}_v(\mathfrak{g}_Q)$, and called the quantum Kac-Moody Lie algebra.

Explicitly, we can describe $\mathcal{U}_v(\mathfrak{g}_Q)$ as the $\mathbb{C}(v)$-algebra generated by $\{E_i, F_i, K_i^{\pm 1}\}_{i \in vertex(Q)}$ satisfy the relations:

$$K_i K_j = K_j K_i \tag{1}$$

$$K_i E_j K_i^{-1} = v^{c_{ij}} E_j \tag{2}$$

$$K_i F_j K_i^{-1} = v^{-c_{ij}} F_j \tag{3}$$

$$[E_i, F_j] = \delta_{i,j} \frac{K_i - K_i^{-1}}{v - v^{-1}} \tag{4}$$

$$\sum_{l=0}^{1-c_{ij}} \binom{1-c_{ij}}{l}_v E_i^l E_j E_i^{1-c_{ij}-l} = 0 \tag{5}$$

$$\sum_{l=0}^{1-c_{ij}} \binom{1-c_{ij}}{l}_v F_i^l F_j F_i^{1-c_{ij}-l} = 0. \tag{6}$$

The subalgebra generated by the E_i's is denoted by $\mathcal{U}_v(\mathfrak{g}_Q^+)$ and is called the positive part of $\mathcal{U}_v(\mathfrak{g}_Q)$.

Suppose the product on the Hall algebra $\mathcal{H}_Q$ is twisted as follows:

$$[M] * [N] := q^{\langle M,N\rangle/2}[M] \cdot [N],$$

where the additive Euler form $\langle \cdot, \cdot \rangle$ is defined via:

$$\langle M, N\rangle = \dim(\mathrm{Hom}(M, N)) - \dim(\mathrm{Ext}^1(M, N)).$$

Theorem (Ringel, Green)

Specializing at $v = q^{1/2}$, there exists an injective (Hopf) algebra homomorphism:

$$\Phi_Q : \mathcal{U}_v(\mathfrak{g}_Q^+) \to \mathcal{H}_Q,$$

where we use the twisted product on the Hall algebra. Furthermore, the above map is an isomorphism exactly when Q is an ADE quiver (with arbitrary orientation).

A limitation of the previous construction is that it only produces 'half' of the quantum group. Algebraically, the whole quantum group can be obtained from either of its 'halves' via a Drinfeld double construction. Morally, this Drinfeld double should be some invariant of a 2-periodic version of the derived category $D^b(\mathcal{C})$.

An approach by Bridgeland: Let $\mathcal{P}_2(\mathcal{C})$ be the category of 2-periodic complexes of projective objects in $\mathcal{C}$. Objects of this category can be viewed as $\mathcal{C}$-valued representations of the quiver:

$$\alpha \underset{e'}{\overset{e}{\rightleftarrows}} \beta \ ,$$

satisfying the relations: $ee' = e'e = 0$.

Define the algebras:

$$\mathcal{DH} := \mathcal{H}_{\mathcal{P}_2(\mathcal{C})}[[M_\bullet]^{-1} : M_\bullet \text{ is acyclic}].$$

$$\mathcal{DH}_{red} := \mathcal{DH}/([M_\bullet] - 1 : M_\bullet \text{ is acyclic and shift invariant}).$$

Theorem (Bridgeland)

Specializing at $v = q^{1/2}$, there exists an injective algebra homomorphism:

$$\Phi_Q : \mathcal{U}_v(\mathfrak{g}_Q) \to \mathcal{DH}_{red}(\mathcal{C}).$$

Furthermore, the above map is an isomorphism exactly when Q is an ADE quiver (with arbitrary orientation).

One could try to pursue the above result in the category of 2-periodic bicomplexes instead:

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & \uparrow & & \uparrow & & \\ \dots & \longrightarrow & W & \xrightarrow{e'} & V & \longrightarrow & \dots \\ & & \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle f'} & & \\ \dots & \longrightarrow & V & \xrightarrow[e]{} & W & \longrightarrow & \dots \\ & & \uparrow & & \uparrow & & \\ & & \vdots & & \vdots & & \end{array}$$

In terms of a quiver, the above category can be thought of as representations of the following quiver:

$$Q_{Rud} := \bullet \begin{array}{c} \xrightarrow{e} \\ \xrightarrow{f} \\ \xleftarrow{e'} \\ \xleftarrow{f'} \end{array} \bullet$$

with the following relations R:

$$ee' = ff' = 0, e'e = f'f = 0$$

$$ef' = fe', e'f = f'e.$$

Remark

The above quiver and relations show up in the description of the blocks of the category of restricted representations of the Lie algebra $\mathfrak{sl}_2$ *over positive characteristic.*

Shifted quantum affine algebras

The quiver

$$Q_{Rud} := \quad \bullet \; \begin{array}{c} \xrightarrow{e} \\ \xrightarrow{f} \\ \xleftarrow{e'} \\ \xleftarrow{f'} \end{array} \; \bullet$$

can be thought of as a doubled version of the Kronecker quiver:

$$Q_K := \quad \bullet \; \begin{array}{c} \xrightarrow{e} \\ \xrightarrow{f} \end{array} \; \bullet$$

Thus, one might expect the Hall algebra of Q_{Rud} to be a doubled version of the Hall algebra of Q_K.

Definition (Finkelberg, Tsymbaliuk)

The shifted quantum affine algebra $\mathcal{U}_v(\widehat{\mathfrak{sl}_2})_{b_1,b_2}$ is isomorphic to the quantum for $b_1, b_2 \in \mathbb{Z}$ is the $\mathbb{C}(v)$-algebra generated by elements $E_l, F_l, H_n, K^{\pm 1}, C^{\pm 1/2}$ for $l \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\}$ satisfying:

$$C^{1/2} \text{ is central}$$

$$KE_kK^{-1} = v^2 E_k, KF_kK^{-1} = v^{-2}F_k, KH_nK^{-1} = H_n$$

$$E_{k+1}E_l - v^2 E_l E_{k+1} = v^2 E_k E_{l+1} - E_{l+1}E_k$$

$$F_{k+1}F_l - v^{-2} F_l F_{k+1} = v^{-2} F_k F_{l+1} - F_{l+1}F_k$$

$$[H_l, E_k] = \frac{[2l]}{l} C^{-|l|/2} E_{k+l}$$

$$[H_l, F_k] = \frac{-[2l]}{l} C^{|l|/2} F_{k+l}$$

$$[H_l, H_k] = \delta_{l,-k} \frac{[2l]}{l} \frac{C^l - C^{-l}}{v - v^{-1}}$$

Definition (Contd.)

$$[E_k, F_l] = \frac{C^{(k-l)/2}\Psi_{k+l} - C^{(l-k)/2}\Phi_{k+l}}{v - v^{-1}}$$

where the elements Ψ_k and Φ_k are defined via the following generating series:

$$u^{b_1} \sum_{k \geq -b_1} \Psi_k u^k = K \exp\left((v - v^{-1}) \sum_{k=1}^{\infty} H_k u^k \right)$$

$$u^{b_2} \sum_{k \geq -b_2} \Phi_{-k} u^k = K^{-1} \exp\left(-(v - v^{-1}) \sum_{k=1}^{\infty} H_{-k} u^k \right).$$

The shifted quantum affine algebra $\mathcal{U}_v(\widehat{\mathfrak{sl}_2})_{1,1}$ turns out to be the relevant object for describing the Hall algebra $\mathcal{H}$ of $\mathrm{Rep}(Q_{Rud}/R)$. Define the twisted Hall algebra $\mathcal{H}_{tw}$ which has the multiplication:

$$[V] * [W] := q^{-\det(\dim(V),\dim(W))/2}[V] \cdot [W],$$

for any $V, W \in \mathrm{Ob}(\mathrm{Rep}(Q_{Rud}/R))$.

Theorem (G.-Samuelson)

There exists an injective algebra homomorphism:

$$\Phi : \mathcal{U}_v(\widehat{\mathfrak{sl}_2})_{1,1} \to \mathcal{H}_{tw},$$

when v is specialized to $q^{1/2}$. The image of this map is exactly the spherical subalgebra of the Hall algebra.

Idea of Proof

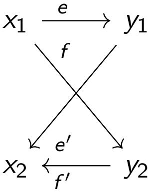
Theorem

All the indecomposable objects in $\mathrm{Rep}(Q_{Rud}/R)$ *can be classified as:*

- *Representations where* $e' = f' = 0$
- *Representations where* $e = f = 0$
- *Two exceptions* M *and* M'*(described below) that are both injective and projective in* $\mathrm{Rep}(Q_{rud}/R)$

Let $V = \mathbb{F}_q\langle x_1, x_2\rangle$ and $W = \mathbb{F}_q\langle y_1, y_2\rangle$ be two-dimensional vector spaces.

$M :=$



The representation M' is obtained from M by flipping along the vertical axis.

We describe what the homomorphism Φ maps the generators of $\mathcal{U}_v(\widehat{\mathfrak{sl}_2})_{1,1}$ to:

$$C^{1/2} \mapsto [M']^{1/4} * [M]^{-1/4}$$

$$K \mapsto [M']^{-1/4} * [M]^{-1/4}$$

For the generators E_l, F_l and H_l, map them to representations where:

- $e' = f' = 0$ if $l \geq 0$
- $e = f = 0$ if $l \leq 0$.

Theorem

Indecomposable representations of the Kronecker quiver over a field F are of three types:

- *Pre-projective representations:*

$$P_n := F^n \overset{e}{\underset{f}{\rightrightarrows}} F^{n+1},$$

where e is the inclusion into the first n coordinates of F^{n+1}, f is the inclusion into the last n.

- *Pre-injective representations:*

$$I_n := F^{n+1} \overset{e}{\underset{f}{\rightrightarrows}} F^n,$$

where e is the projection onto the first n coordinates of F^{n+1}, f is the projection onto the last n.

Theorem (Contd.)

- *Regular representations:*

$$F^n \overset{e}{\underset{f}{\rightrightarrows}} F^n \ ,$$

where one of e or f is the identity map, and the other is given by an indecomposable $n \times n$ matrix over F.

When $l \geq 0$, the homomorphism Φ 'roughly' maps:

- E_l to $[I_l]$
- F_l to $[P_l]$
- H_l to a weighted average of regular representations having dimension (l, l).

Remark

Regular representations come in families that are indexed by irreducible polynomials over $\mathbb{F}_q$,and the subalgebra generated by representations in any such family is a quantum Heisenberg algebra.

The inclusion of the Kronecker quiver into the Rudakov quiver induces a functor:

$$F : \mathrm{Rep}(Kron) \to \mathrm{Rep}(Q_{Rud}/R),$$

which is given by extension by zero on e' and f'.This functor is fully faithful and exact and induces a map between the Ext groups for any $A, B \in \mathrm{Ob}(\mathrm{Rep}(Kron))$:

$$\overline{F} : \mathrm{Ext}^1(A, B) \to \mathrm{Ext}^1(F(A), F(B)).$$

While the functor F does not induce a morphism of Hall algebras,we do have the following:

Proposition

Then, the map $\overline{F}$ is an isomorphism, unless A is a pre-projective representation and B is a pre-injective representation.Furthermore, if the map is not an isomorphism, then either A or B must be a simple representation.

Thank You!