13/06/22

Motivation: orbit method

$G \longrightarrow$ connected Lie group

Goal: Classify unitary G-irreps

Kirillov (1961) Orbit method

$\mathfrak{g} = \mathrm{Lie}(G)$ $\qquad G \curvearrowright \mathfrak{g}, \mathfrak{g}^*$

adjoint, coadjoint reps.

Theorem: (Kirillov) If G is nilpotent, simply connected,

$$\{\text{unitary } G\text{-irreps.}\}/\text{iso.} \xrightarrow{\sim} \{G\text{-orbits in } \mathfrak{g}^*\}$$

natural

Why should coadjoint orbits appear in this classification?

Answer: Quantization (connection between classical and quantum mechanical systems)

	Quantum	Classical
Phase Space	Hilbert Space	I

Symmetry	Unitary representation	II
Most symmetric	The rep. is irreducible	G acts transitively

I $\longrightarrow$ Manifold M with Poisson bracket:

$\mathbb{R}$-linear $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$

s.t.
- $\{\cdot,\cdot\}$ is a Lie bracket.
- Leibniz: $\{fg, h\} = \{f,h\}g + \{g,h\}f$

to give $\{\cdot,\cdot\} \longleftrightarrow$ bivector field P with certain properties

s.t. $\{f,g\} := \langle P, df \wedge dg \rangle$

Special case: ω is a symplectic form on M
($d\omega = 0$, non-degenerate)

$\rightsquigarrow$ bivector $\omega^{-1} \rightsquigarrow \{\cdot,\cdot\}$
$\hookrightarrow$ non-degenerate Poisson structure.

II $\longrightarrow$ Lie group $G \curvearrowright M$ preserving $\{\cdot,\cdot\}$ and "Hamiltonian"

$G \curvearrowright M \rightsquigarrow G$-equiv. linear map

$$\mathfrak{g} \longrightarrow \mathrm{Vect}(M)$$
$$\xi \longmapsto \xi_M$$

Def^n: The classical comoment map is a G-equivariant linear map $\phi : \mathfrak{g} \longrightarrow C^\infty(M)$ s.t.
$$\xi_M = \{\phi(\xi), \cdot\} \quad \forall \xi \in \mathfrak{g}$$

Dually, we get a moment map $\mu : M \longrightarrow \mathfrak{g}^*$
$$\langle \mu(m), \xi \rangle := [\phi(\xi)](m).$$

- $G \curvearrowright M$ is Hamiltonian if $\{\cdot,\cdot\}$ is G-invariant and we've fixed a comoment map.

(If G is connected, the G-invariance follows from the existence of the comoment map.)

Exercise 1: Show that ϕ is a Lie algebra homomorphism, i.e. $\phi([\xi, \eta]) = \{\phi(\xi), \phi(\eta)\} \quad \forall \xi, \eta \in \mathfrak{g}$.

Example: $\mathfrak{g}^*$ is Poisson with unique $\{\cdot,\cdot\}$ s.t.
$$\{\xi, \eta\} := [\xi, \eta] \quad \forall \xi, \eta \in \mathfrak{g} \subseteq C^\infty(\mathfrak{g}^*).$$
Then, this action $G \curvearrowright \mathfrak{g}^*$ is Hamiltonian with
$$\phi(\xi) = \xi \quad (\text{equivalently, } \mu = \mathrm{Id})$$

Transitive Hamiltonian actions

Exercise 2 : $\alpha \in \mathfrak{g}^* \rightsquigarrow M = G\alpha$. Want to construct
$\mathcal{P}_{G\alpha} \in \Gamma(\Lambda^2 T_M)$.

i.e. 1) $\mathcal{P} \in \Gamma(\Lambda^2 T_{\mathfrak{g}^*})$. Then, $\mathcal{P}_{G\alpha} \in \Lambda^2 T_{G\alpha}$ and is non-degenerate there, uniquely extends to a G-invariant $\mathcal{P}_{G\alpha} \in \Gamma(\Lambda^2 T_{G\alpha})$, which is Poisson.

2) $\omega_\alpha := \mathcal{P}_{G\alpha}^{-1}$ is the unique G-equivariant 2-form satisfying $\omega_\alpha(\xi\alpha, \eta\alpha) = \langle \alpha, [\xi, \eta] \rangle$.

3) $G \curvearrowright G\alpha$ is Hamiltonian with $\mu : G\alpha \hookrightarrow \mathfrak{g}^*$ the inclusion map.

Exercise 3 : Let M be a Poisson manifold with transitive Hamiltonian $G \curvearrowright M$. Then,

1) $\mathrm{Im}(\mu) \subseteq \mathfrak{g}^*$ is a single orbit
2) $\mu : M \longrightarrow \mathrm{Im}(\mu)$ is a cover and is Poisson, i.e. $\mu^* : C^\infty(\mathrm{Im}(\mu)) \longrightarrow C^\infty(M)$ intertwines $\{\cdot, \cdot\}$.
3) The Poisson structure on M is non-degenerate and μ is a symplectomorphism.

<u>Conclusion</u> : Transitive Hamiltonian actions $\longleftrightarrow$ G-equivariant covers of coadjoint orbits

• Exercise* : Given the Hamiltonian action of a Lie group on a simply connected manifold, then it can be seen as an action of a central extension.

So, the orbit method predicts a connection:

$$\left\{ \begin{array}{c} \text{Equivariant } G\text{-covers} \\ \text{of coadjoint orbits} \end{array} \right\} \overset{\text{Quantization}}{\dashrightarrow} \left\{ \begin{array}{c} \text{unitary } G\text{-} \\ \text{irreps} \end{array} \right\}$$

- If G is nilpotent, this is a bijection.
  (There are no complex G-covers.)
- If G is semisimple, this isn't a bijection.
  Eg: If G is compact,
  unitary reps. = finite dim. reps.
  (classified by highest wt.s)

Also, as G is semisimple $\Rightarrow$ (co-)adjoint orbits $\overset{\sim}{\longleftrightarrow}$ Weyl chambers

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<u>Filtered quantizations</u> (of algebras)

Many algebras of interest for geometric representation theory arise as filtered quantizations.

Setting: A is f.g. commutative $\mathbb{C}$-algebra s.t.
1) A is $\mathbb{Z}_{\geq 0}$-graded $A = \bigoplus_{i=0}^{\infty} A_i$ (as vector spaces)

s.t. $A_i A_j \subseteq A_{i+j}$

2) Poisson compatibility $\{\cdot,\cdot\} : A \times A \longrightarrow A$

3) $\exists\, \alpha \in \mathbb{Z}_{>0}$ s.t. degree$(\{\cdot,\cdot\}) = -\alpha$ i.e.

$$\{A_i, A_j\} \subseteq A_{i+j-\alpha}$$

Examples :

1) $\mathfrak{g} \longrightarrow$ f.d. Lie algebra, $A = S(\mathfrak{g})$ $(= \mathbb{C}[\mathfrak{g}^*])$.
$\exists!$ $\{\cdot,\cdot\}$ on A s.t. $\{\xi, \eta\} = [\xi, \eta]$ with usual grading and $\alpha = 1$.

2) $V \longrightarrow$ symplectic vector space with form ω.
$A = S(V)$ $(= \mathbb{C}[V^*])$, usual grading.
$\exists!$ $\{\cdot,\cdot\}$ s.t. $\{u, v\} = \omega(u,v)$ $\forall\, u, v \in V$
and $\alpha = 2$.

Def^n : (Filtered quantization of A) This is a pair $(\mathcal{A}, i)$
where : • $\mathcal{A}$ is an associative $\mathbb{C}$-algebra with an algebra filtration $\mathcal{A} = \bigcup_{i=0}^{\infty} \mathcal{A}_{\leq i}$ (as vector spaces) s.t.
$1 \in \mathcal{A}_{\leq 0}$ and $\mathcal{A}_{\leq i} \mathcal{A}_{\leq j} \subseteq \mathcal{A}_{i+j}$ $\forall\, i, j$
and degree$([\cdot,\cdot]) \leq -\alpha$ i.e.

$$[\mathcal{A}_{\leq i}, \mathcal{A}_{\leq j}] \subseteq \mathcal{A}_{i+j-\alpha}$$

($\rightsquigarrow$ $\mathrm{gr}\,\mathcal{A} = \bigoplus_{i=0}^{\infty} \mathcal{A}_{\leq i} / \mathcal{A}_{\leq i-1}$ which is a Poisson algebra with

$$\{\underset{\mathcal{A}_{\leq i}}{\overset{\cap}{a}} + \mathcal{A}_{\leq i-1}, \underset{\mathcal{A}_{\leq j}}{\overset{\cap}{b}} + \mathcal{A}_{\leq j-1}\} = [a,b] + \mathcal{A}_{\leq i+j-\alpha-1})$$

- $i : \mathrm{gr}(\mathcal{A}) \longrightarrow A$ is an isomorphism of graded Poisson algebras.

Def^n: An isomorphism of filtered quantizations $(\mathcal{A}, i)$, $(\mathcal{A}', i')$ is a filtered algebra isomorphism $\varphi : \mathcal{A} \to \mathcal{A}'$ s.t.

$$\begin{tikzcd} \mathrm{gr}\,\varphi : \mathrm{gr}\,\mathcal{A} \arrow[rr, "\sim"] \arrow[dr, "i"'] & \arrow[d, phantom, "\circlearrowright"] & \mathrm{gr}\,\mathcal{A}' \arrow[dl, "i'"] \\ & A & \end{tikzcd}$$

Examples:

1) $A = S(\mathfrak{g})$

Then, $\mathcal{A} = \mathcal{U}(\mathfrak{g})$ is a filtered quantization by PBW theorem.

2) $A = S(V)$

Then, $\mathcal{A} = W(V) = T(V) \Big/ \Big(u \otimes v - v \otimes u - \omega(u,v)\Big)_{u,v \in V}$ (the Weyl algebra)

is the unique filtered quantizations. (Exercise.)

Problem: Given A, classify its filtered quantizations upto isomorphism.

$\rightsquigarrow$ Cannot solve without additional restrictions on A.

Restriction = "symplectic singularities"

We'll care about A coming from "nilpotent orbits" in s.s. Lie algebras. The restriction holds in that setting.

<u>Nilpotent orbits (in s.s. Lie algebras)</u>

$G \longrightarrow$ connected, s.s. algebraic group / $\mathbb{C}$
$\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{g} \underset{G}{\simeq} \mathfrak{g}^*$ via the Killing form
So, adjoint orbits = coadjoint orbits.
$G \curvearrowright \mathfrak{g}$ with all orbits symplectic (algebraic) varieties.
$\Downarrow$
even-dim.

<u>Defn</u>: $\xi \in \mathfrak{g}$ is nilpotent if it is represented by a nilpotent operator in some (any) faithful $\mathfrak{g}$-rep.

<u>Remark</u>: 1) If $\mathfrak{g}$ is classical ($= \mathfrak{sl}_n, \mathfrak{so}_n, \mathfrak{sp}_n$), then nilpotent elements = nilpotent matrices.
2) For $g \in G$, $\xi \in \mathfrak{g}$,
ξ is nilpotent $\iff$ $Ad(g)\,\xi$ is nilpotent.
So, we can talk about nilpotent orbits.

Question: How to classify nilpotent orbits?
$\Downarrow$?
Lie alg. homo. $\mathfrak{sl}_2 \longrightarrow \mathfrak{g}$.

Def^n: An sl_2-triple in $\mathfrak{g}$ is a triple $(e,h,f) \in \mathfrak{g}$ satisfying the defining relations of sl_2, i.e. the map $\phi : sl_2 \longrightarrow \mathfrak{g}$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longmapsto e, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longmapsto h, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \longmapsto f$$

is a Lie algebra homomorphism.

Exercise: e ($\&$ f) are nilpotent.

Theorem 1: (Jacobson-Morozov) $\forall$ nilpotent $e \in \mathfrak{g}$ can be included into an sl_2-triple.

Theorem 2: (Kostant) If (e,h,f), (e,h',f') are sl_2-triples, then $\exists\, g \in G$ s.t.

$$Ad(g)e = e, \quad Ad(g)h = h', \quad Ad(g)f = f'.$$

Corollary: The map

$$\left\{ \begin{matrix} \text{conjugacy classes} \\ \text{of } sl_2 \to \mathfrak{g} \end{matrix} \right\} \longrightarrow \{\text{nilpotent orbits}\}$$

$$(e,h,f) \longmapsto e$$

is a bijection.

Example: $\mathfrak{g} = sl_n$ (nilpotent orbits $\longleftrightarrow$ partitions of n)

by taking Jordan type

$\mathfrak{sl}_2 \longrightarrow \mathfrak{g}$ are just n-dim $\mathfrak{sl}_2$-reps.

Conjugacy classes = isom classes of $\mathfrak{sl}_2$-reps

n-dim $\mathfrak{sl}_2$-rep. is $\bigoplus_{i=1}^{k} V(d_i)$

$\hookrightarrow$ d_i-dim rep. of $\mathfrak{sl}_2$.

$\overset{\sim}{\longleftrightarrow}$ partitions $((d_1, \ldots, d_k))$

$e \in \mathfrak{sl}_2$ acts by one Jordan block on every irrep.

Hence, the above corollary reduces to the usual linear algebraic classification.

<u>Exercise</u>: If (e, h, f) and (e, h, f') are $\mathfrak{sl}_2$-triples, then $f = f'$.

Take $G = Sp_n$ or O_n

O_n isn't simply connected

$O_n / SO_n \simeq \mathbb{Z}/2\mathbb{Z}$.

<u>Prop</u>^{n}: Nilp. G-orbits in $\mathfrak{g}$

$\updownarrow$ 1:1

Partitions of n where every even (for O_n), odd (for Sp_n) part has even multiplicity.

<u>Remark</u>: A nilp. orbit for O_n splits into two SO_n-orbits $\iff$ all parts of the corresponding partition are even.

Next, we discuss the algebras $\mathbb{C}[\mathbb{O}]$ - regular (a.k.a.

polynomial functions), $\mathbb{O}$ is a nilp. G-orbit in $\mathfrak{g}$.

<u>Theorem</u>: $\mathbb{C}[\mathbb{O}]$ is finitely generated, graded and has Poisson bracket of degree -1.

<u>Proof</u> (sketch): $\mathbb{O} \to$ symplectic variety
$\rightsquigarrow \mathbb{C}[\mathbb{O}]$ is equipped with a Poisson bracket.

Fact 1: # of nilpotent orbits in $\mathfrak{g}$ is finite.

(<u>Exercise</u>: # of conjugacy classes of Lie alg. homo $\mathfrak{g} \to \mathfrak{g}'$ for arbit s.s. Lie algebras is finite.)

It is clear that $\mathbb{C}[\bar{\mathbb{O}}]$ is finitely generated.

<u>Exercise</u>: The nilpotent cone $\mathcal{N} = \{\xi \in \mathfrak{g} : \xi \text{ is nilp.}\}$ is Zariski closed.

$\bar{\mathbb{O}} \setminus \mathbb{O}$ consists of nilpotent orbits, of which there are finitely many and all have an even dimension.
$\Rightarrow \operatorname{codim}_{\bar{\mathbb{O}}} (\bar{\mathbb{O}} \setminus \mathbb{O}) \geq 2$.

Fact 2: Let X be an affine irreducible variety. Let $X^\circ \subseteq X$ be a smooth open subvariety s.t.

$\operatorname{codim}_X (X \setminus X^\circ) \geq 2$. (Eg: $X = \bar{\mathbb{O}}$, $X^\circ = \mathbb{O}$).)
Then, $\mathbb{C}[X^\circ]$ is the normalization of $\mathbb{C}[X]$ $\Rightarrow$
$\mathbb{C}[X^\circ]$ is finitely generated.

Hence, $\mathbb{C}[\mathbb{O}]$ is finitely generated.

Next, there is an action of $\mathbb{C}^\times$ on $\mathfrak{g}$ by dilations.
$$t \cdot \xi = t^{-1} \xi.$$
Then, $\mathbb{O}$ is $\mathbb{C}^*$-stable. (follows from the classification in classical types).

$\mathbb{C}^* \curvearrowright \mathbb{O} \rightsquigarrow$ grading on $\mathbb{C}[\mathbb{O}]$.

<u>Exercise</u>: The degree of the Poisson bracket is -1.
(Follows by following the def^n. of $\{\cdot, \cdot\}$ in terms of the <u>symplectic form</u>.)

<u>14/06/22</u>

<u>Equivariant covers of nilpotent orbits</u>

Let $e \in \mathfrak{g}$ be nilpotent and $H = Z_G(e)$.
An equivariant cover of $G \cdot e = G/H$ is G/H' s.t.
$H \supseteq H' \supseteq H^\circ$ are subgroups of finite index.

Hence,

equivariant covers of $G\cdot e$ $\longleftrightarrow$ subgroups of $Z_G(e)/Z_G(e)^\circ$

Exercise: 1) $Z_G(e) = \underbrace{Z_G(e,h,f)}_{\text{reductive}} \ltimes \underbrace{\text{unipotent}}_{\text{connected}}$

2) $Z_G(e)/Z_G(e)^\circ \simeq Z_G(e,h,f)/Z_G(e,h,f)^\circ$

Prop^n: $G = SL_n, O_n, Sp_n$. Let $\mathbb{O} = Ge \subseteq \mathfrak{g}$ be a nilpotent orbit with partition $(1^{d_1}\dots n^{d_n})$.
(superscripts are multiplicities)

1) $G = SL_n \rightsquigarrow Z_G(e,h,f) \simeq \{(g_1,\dots,g_n) \in \prod_{i=1}^{n} GL(d_i)$ s.t. $\det(g_i)^i = 1\}$

Then, $Z_G(e,h,f)/Z_G(e,h,f)^\circ$

$$\simeq \frac{\mathbb{Z}}{GCD(i : d_i \neq 0)}$$

2) $G = O_n$ or $Sp_n \rightsquigarrow Z_G(e,h,f) = \prod_{i=1}^{n} G_i$, where

$$G_i = \begin{cases} O_{d_i} & \text{if } G = O_n \text{ and } i \text{ is odd} \\ & \text{or} \\ & G \text{ is } Sp_n \text{ and } i \text{ is even} \\ Sp_{d_i} & \text{otherwise} \end{cases}$$

So, $Z_G(e,h,f) / Z_G(e,h,f)^\circ \simeq (\mathbb{Z}/2\mathbb{Z})^a$,
where a = # of O factors

$$= \begin{cases} \# \{\text{odd } i \text{ with } d_i \neq 0\} & \text{for } G = O_n \\ \# \{\text{even } i \text{ with } d_i \neq 0\} & \text{for } G = Sp_n \end{cases}$$

<u>Example</u>: $G = Sp_n$, $\mathbb{O}$ corresponding to $(2, 1^{n-2})$

$$Z_G(e)/Z_G(e)^\circ \simeq \mathbb{Z}/2\mathbb{Z}$$

Then, a 2-fold cover of $\mathbb{O}$ is given by $\mathbb{C}^n \setminus \{0\}$.

<u>Theorem</u>: Let $\tilde{\mathbb{O}}$ be G-equivariant cover of $\mathbb{O}$. Then, $\mathbb{C}[\tilde{\mathbb{O}}]$ is finitely generated, graded, Poisson.

<u>Sketch of proof</u>:

- It is Poisson because $\tilde{\mathbb{O}}$ is symplectic. (Because $\mathbb{O}$ is symplectic).
- The morphism $\tilde{\mathbb{O}} \longrightarrow \overline{\mathbb{O}}$ has finite fibres

So, there exists a Stein factorization:

$$\begin{tikzcd} \tilde{\mathbb{O}} \arrow[rr] \arrow[dr] & & \overline{\mathbb{O}} \\ & X \arrow[ur] & \end{tikzcd}$$

X = Spec(integral closure of $\mathbb{C}[\overline{\mathbb{O}}]$ in the fraction field of $\mathbb{C}[\tilde{\mathbb{O}}]$).

Then, $X \longrightarrow \overline{\mathbb{O}}$ is finite and $\tilde{\mathbb{O}} \underset{\text{open}}{\hookrightarrow} X$.

$\rightsquigarrow$ $X \setminus \tilde{\mathcal{O}} \subseteq X$ has codim. 2.

(as $\operatorname{codim}_{\bar{\mathcal{O}}} \bar{\mathcal{O}} \setminus \mathcal{O} \geq 2$)

$\tilde{\mathcal{O}}$ is smooth, and so, (by a fact from last time) $\mathbb{C}[\tilde{\mathcal{O}}] = \mathbb{C}[X]$, which is f.g.

To show that $\mathbb{C}[\tilde{\mathcal{O}}]$ is graded, we note that it's possible to lift $\mathbb{C}^\times \curvearrowright \bar{\mathcal{O}}$ to $\tilde{\mathcal{O}}$ after rescaling.
(That is, $z \cdot \xi = \underline{z^{-d}\xi \text{ for suitable } d}$).

Singular Symplectic Varieties

<u>Defn</u>: If X is a smooth algebraic variety, then X is symplectic if it has a symplectic form.

$\Rightarrow$ X is Poisson, $\mathcal{O}_X$ has $\{\cdot,\cdot\}$.

Beauville (2000): Notion of symplectic for singular Poisson varieties.

<u>Defn</u>: Let X be a Poisson variety. We say X is (singular) symplectic (has symplectic

singularities) if:

i) X is normal (and X is irreducible)

ii) The Poisson structure on X^{reg} is non-degenerate.

Let ω^{reg} be the symplectic form on X^{reg}.

iii) $\exists$ resolution of singularities $\pi: Y \longrightarrow X$ (i.e., Y is smooth, π is proper, birational) s.t. $\pi^* \omega^{reg}$ (a 2-form on $\pi^{-1}(X^{reg})$) extends to a (regular) 2-form on Y.

Remarks:

- Beauville showed that if (iii) holds for some resolution, it is true for all resolutions.

- If $\pi^* \omega^{reg}$ extends to a non-degenerate form on Y, we call Y a symplectic resolution of X.

Examples:

1) Symplectic quotient singularities

$V \longrightarrow$ f.d. symplectic vector space with form ω.

Let $\Gamma \subseteq Sp(V)$ be a finite subgroup.

Construct $X = V/\Gamma = \mathrm{Spec}\,(\mathbb{C}[V]^\Gamma)$.

Γ preserves $\{\cdot,\cdot\}$ on $\mathbb{C}[V]$

$\Rightarrow$ $\mathbb{C}[V]^\Gamma$ is a Poisson subalgebra on $\mathbb{C}[V]$.

i) X is normal.

ii) $X^{reg} = \{$free Γ-orbits in $V\}$ $\rightsquigarrow$ unramified

Let $\eta : V \longrightarrow V/\Gamma$.

Consider $\eta : \eta^{-1}(X^{reg}) \longrightarrow X^{reg}$.

Then, ω^{reg} is obtained by descent of ω from $\eta^{-1}(X^{reg})$.

iii) $\longrightarrow$ checked by Beauville.

Sometimes, V/Γ has symplectic resolutions.

a) $\dim V = 2$. $\Gamma \subseteq SL_2(\mathbb{C})$

$\Rightarrow X = \mathbb{C}^2/\Gamma$.

Take Y to be the unique minimal resolution.

(For example, if $\Gamma = \{\pm 1\}$, $Y = T^*\mathbb{P}^1$)

Then, Y is symplectic.

Remark: This exhausts all dimension 2 symplectic singularities.

b) $V = (\mathbb{C}^2)^{\oplus n} \curvearrowleft S_n \rightsquigarrow X = V/\Gamma$

$$\begin{array}{c} X \\ \uparrow \text{symplectic resolution} \\ Y = \mathrm{Hilb}_n(\mathbb{C}^2) \end{array}$$

2) $X = \operatorname{Spec} \mathbb{C}[\tilde{\mathbb{O}}]$.

Theorem: X is singular symplectic

- $\tilde{\mathbb{O}} = \mathbb{O}$ ($\subset \mathfrak{g}$): Panyushev, Hinich
- General case follows from here by some algebraic geometry.

When does X admit a symplectic resolution Y and what does it look like?

Answer: Y is always $T^*(G/P)$.

Here, P is a parabolic subgroup of G. That is, equivalently:

- P containts a Borel
- G/P is projective.

We have a decomposition $P = L \ltimes U$

L: connected, reductive (Levi subgroup)

U: unipotent

For $G = SL_n$, we pick a composition $n = n_1 + \ldots + n_k$

$$P = \left\{ \begin{pmatrix} \boxed{\begin{matrix} \times & \times \\ \times & \times \end{matrix}} & \times & \times & \times \\ 0 \quad 0 & \boxed{\times} & \times & \times \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} & \boxed{\begin{matrix} \times & \times \\ \times & \times \end{matrix}} \end{pmatrix} \right\} \longleftrightarrow 5 = 2 + 1 + 2$$

$$L = \left\{ \begin{pmatrix} \square & & 0 \\ & \square & \\ 0 & & \square \end{pmatrix} \right\} \leftarrow \text{block diagonal}$$

$$\mathcal{U} = \operatorname{Ker}(P \twoheadrightarrow L)$$

Let $\mathfrak{n} = \operatorname{Lie}(\mathcal{U})$.

Then, $T^*(G/P) = G \times^P \underset{\substack{\uparrow \\ \text{cotangent space at } eP.}}{(\mathfrak{g}/\mathfrak{p})^*}$

$$= (G \times (\mathfrak{g}/\mathfrak{p})^*)/P$$

where P acts via $p \cdot (g, \alpha) = (gp^{-1}, p \cdot \alpha)$

$\underset{\substack{\uparrow \\ P\text{-orbit of } (g,\alpha)}}{[g,\alpha]} \in G \times^P (\mathfrak{g}/\mathfrak{p})^*$

Then, we have a Hamiltonian action

$$G \curvearrowright T^*(G/P).$$

Note that $\mathfrak{p}^\perp = \mathfrak{n}$ (w.r.t. the Killing form).

$\Rightarrow (\mathfrak{g}/\mathfrak{p})^* = \mathfrak{n}$

So, $T^*(G/P) = G \times^P \mathfrak{n} \xrightarrow{\mu} \mathfrak{g}^* \simeq \mathfrak{g}$

where μ is the moment map.

$$[g, \alpha] \longmapsto \mathrm{Ad}(g)\alpha$$

We can check that this is well-defined.

Exercise: μ is proper. (In fact, it is projective.)

Hence, $\mathrm{Im}(\mu)$ is closed.

Let's describe $\mathrm{Im}(\mu) = G\mathfrak{n} \subseteq \mathcal{N}$ nilp. cone.

(as $\mathfrak{n}$ consists of nilpotent elements.)

$\mathrm{Im}(\mu)$ is irreducible because $T^*(G/P)$ is.

$\uparrow$

union of finitely many orbits.

$\Rightarrow \exists$ a nilpotent orbit $\mathbb{O}_P \subset \mathfrak{g}$ s.t. $\mathrm{Im}(\mu) = \overline{\mathbb{O}_P}$

Fact (to be seen later): $\dim(\mathbb{O}_P) = \dim(T^*(G/P))$

$\Rightarrow \mu$ is finite-to-one generically.

$\Rightarrow G \curvearrowright T^*(G/P)$ has an orbit $\tilde{\mathbb{O}}_P$ of dim. equal to $\dim(T^*(G/P))$.

Then, $\tilde{\mathbb{O}}_P$ is open and is a cover of $\mathbb{O}_P$.

$\downarrow$

called Richardson orbit

Let $X = \mathrm{Spec}(\mathbb{C}[\tilde{\mathbb{O}}_P])$.

Exercise: We have a commutative diagram

$$\begin{tikzcd} T^*(G/P) \arrow[r] \arrow[d, dashed, "\pi"'] \arrow[dr] & \mathfrak{g} \\ X \arrow[r] & \overline{\mathcal{O}}_P \arrow[u] \end{tikzcd}$$

s.t. π is a symplectic resolution.

Examples:

1) $P = B$ Borel, $Y = T^*(G/B)$

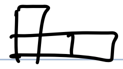
Then, $\mathrm{im}(\mu) = \mathcal{N}$.

$\pi : T^*(G/B) \longrightarrow \mathcal{N}$ is a symplectic resolution, called the Springer resolution.

($\tilde{\mathcal{O}} = \mathcal{N}$, in this case)

2) $G = SL_n$, $\mathcal{O} = \mathcal{O}_\lambda$, where λ is a partition.

Then, we can construct λ^t.

(Eg: $\lambda =$  $\Longleftrightarrow$ $\lambda^t =$ )

Let $n = n_1 + \ldots + n_k$ is a composition obtained from λ^t, in some order.

$\rightsquigarrow$ can construct the corresponding $P \subseteq G$.

Then, $\overline{\mathcal{O}}$ is normal. (Kraft, Procesi)

and $T^*(G/P) \longrightarrow \overline{\mathcal{O}}_\lambda$ is birational, and so, is a symplectic resolution.

3) $\mathfrak{g} = \mathfrak{sp}_4$ and consider the partition $(2,2)$.
Let P_1, P_2 be the semisimple rank 1 parabolics.
Then, $T^*(G/P_i)$ are symplectic resolutions for
$\mathrm{Spec}(\mathbb{C}[\mathbb{O}])$, $\mathrm{Spec}(\mathbb{C}[\tilde{\mathbb{O}}])$.
$\uparrow$ degree 2 cover.

Classification of filtered quantizations

Setting: A is a f.g. commutative graded Poisson algebra with degree of $\{\cdot,\cdot\} = -\alpha$. ($\alpha \in \mathbb{Z}_{>0}$).
s.t. 1) $A_0 = \mathbb{C}$
2) $X = \mathrm{Spec}(A)$ is singular symplectic.

An X satisfying these two conditions is said to have conical symplectic singularities.

Theorem: (Losev, 2016) Suppose A satisfies 1), 2).
Then, $\exists$ f.d. vector space $\mathfrak{h}_X$ and a finite crystallographic reflection group $W_X \curvearrowright \mathfrak{h}_X$ s.t.
linear action

$$\{\text{filtered quantizations of } A\}/\sim \ \xrightarrow{\sim} \ \mathfrak{h}_X / W_X$$

natural

$W_X \longrightarrow$ Namikawa Weyl group.

Example: $\mathfrak{g} \longrightarrow$ s.s. Lie algebra and $X = \mathcal{N}$.
Then, X has conical symplectic singularities.
In that case, $\mathfrak{h}_X = \mathfrak{h}^*$ and $W_X = W$.
($\mathfrak{h} \longrightarrow$ Cartan subalgebra, $W \longrightarrow$ Weyl group)

Construction of quantization:

$$\mathcal{U}(\mathfrak{g}) \supseteq \underset{\overset{\shortparallel}{Z}}{Z\mathfrak{g}} \simeq \mathbb{C}[\mathfrak{h}^*]^W$$

$\hookrightarrow$ Harishchandra isomorphism

(We consider the usual action of W on $\mathbb{C}[\mathfrak{h}^*]$ here, twisting the action doesn't change the algebra.)

Let $\lambda \in \mathfrak{h}^*/W \rightsquigarrow$ maximal ideal in $\underset{\overset{\shortparallel}{Z}}{\mathbb{C}[\mathfrak{h}^*]^W}$

$\rightsquigarrow \mathcal{U}_\lambda := \mathcal{U}\mathfrak{g} / (\mathcal{U}\mathfrak{g} \cdot m_\lambda)$

As $m_\lambda \subseteq Z$, $(\mathcal{U}\mathfrak{g} \cdot m_\lambda)$ is a 2-sided ideal.

Exercise: $\mathcal{U}_\lambda$ is a filtered quantization of $\mathbb{C}[\mathcal{N}]$.

The correspondence in the theorem is

$$\mathcal{U}_\lambda \longleftrightarrow \lambda.$$

Q: 1) How to compute $\mathfrak{h}_X$?
2) How to construct a quantization starting

from a point in $\mathfrak{h}_X$?

Partial answer to 1: Suppose Y is a symplectic resolution of X. Then, $\mathfrak{h}_X = H^2(Y, \mathbb{C})$.

Example: $X = \mathcal{N}$, $Y = T^*(G/B)$.

$$H^2(Y, \mathbb{C}) = H^2(G/B, \mathbb{C})$$

Let G be simply connected and $F \subset G$ be an algebraic group. Then, using the spectral sequence for the cohomology of fiber bundles,

$$H^2(G/F, \mathbb{C}) = H^1(F^0, \mathbb{C})^{F/F^0}$$

Hence, $H^2(G/B, \mathbb{C}) = H^1(B, \mathbb{C}) = \mathfrak{h}^*$, which is exactly what we expected.

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Recall: $A \longrightarrow$ finitely generated commutative Poisson algebra over $\mathbb{C}$. $X = \mathrm{Spec}(A)$. Suppose $A_i = 0 \ \forall i < 0$ and $A_0 = \mathbb{C}$ (i.e. X is conical) and X is singular symplectic.

i) X is normal

ii) X^{reg} is symplectic with form ω^{reg}

iii) For all resolution of singularities $\hat{\pi} : \hat{Y} \to X$, $\hat{\pi}^*(\omega_{reg})$ extends to $\hat{Y}$.

Theorem: $\{\text{filtered quantizations}\}/\sim \xrightarrow{\sim} \mathfrak{h}_X / W_X$

How to compute $\mathfrak{h}_X$?

If $Y \to X$ is a symplectic resolution, we take

$$\mathfrak{h}_X = H^2(Y, \mathbb{C}).$$

$\mathbb{Q}$-factorial terminalizations

In general, we replace Y with a maximal partial Poisson resolution of X, i.e.

partial
- $\pi : Y \to X$ proper, birational, Y may be singular

Poisson
- Y is a Poisson variety, π is a Poisson map i.e. $\forall f, g \in \mathbb{C}[X]$, $\{\pi^*(f), \pi^*(g)\} = \pi^*\{f, g\}$.

Maximal
- If $\pi' : Y' \to Y$ is proper, birational, Poisson, then π' is an isomorphism.

In particular, Y must be normal. Otherwise $\pi' : Y' \to Y$, the normalization morphism, is such that $\exists !$ Poisson structure on Y' making π' a Poisson map. (Kaledin)

Exercise: This Y is singular symplectic.

Hint: Take a resolution $\hat{\pi} : \hat{Y} \longrightarrow Y$ that is an isomorphism over Y^{reg}. Then,

$$\hat{\pi} \circ \pi : \hat{Y} \longrightarrow X \Rightarrow \underbrace{\pi^* \omega_{reg}}_{\text{symplectic}} \text{ extends to } Y^{reg}.$$

(Here, we don't use the maximality of Y.)

Remark: Using this argument, one can show that a symplectic resolution is maximal.

Existence of a maximal, partial, Poisson resolution Y is non-obvious (but true!). Also, it is known that Y has an algebro-geometric characterization: It's "$\mathbb{Q}$-factorial" and "terminal".

"$\mathbb{Q}$-factorial": Given a scheme Z, we have its Picard group $\mathrm{Pic}(Z) = \text{group } \{\text{line bundles on } Z\} / \text{iso}, \otimes$

Defn: Let Z be a normal, irreducible variety. We say Z is $\mathbb{Q}$-factorial if $\mathrm{coker}\left[\mathrm{Pic}(Z) \xrightarrow[res]{} \mathrm{Pic}(Z^{reg})\right]$ is torsion.

Example: Let $\tilde{\mathbb{O}}$ be a G-equivariant cover of a nilpotent orbit $\mathbb{O}$ in $\mathfrak{g}$. Let $X = \mathrm{Spec}\,(\mathbb{C}[\tilde{\mathbb{O}}])$.

- As X is conical, $\mathrm{Pic}(X) = \{0\}$
  (graded Nakayama lemma)

- Next, we compute $\mathrm{Pic}(X^{reg})$. Now, $\tilde{\mathbb{O}} \hookrightarrow X^{reg}$, with complement having codimension ≥ 2. Thus,
$$\mathrm{Pic}(X^{reg}) = \mathrm{Pic}(\tilde{\mathbb{O}}).$$

Now, $\tilde{\mathbb{O}} = G/H$ with G - simply connected
$$\Rightarrow \mathrm{Pic}(G/H) \xleftarrow{\sim} \mathfrak{X}(H) = \mathrm{Hom}(H, \mathbb{C}^\times).$$
So, X is $\mathbb{Q}$-factorial $\Leftrightarrow$ $\mathfrak{X}(H)$ is finite.

$$H \subseteq Z_G(e) = \underbrace{Z_G(e,h,f)}_{\downarrow} \ltimes \text{unipotent group}.$$

This has been computed for G classical.

For $G = SL_n$, $\tilde{\mathbb{O}} \xrightarrow{\sim} \mathbb{O}\ (\subseteq \mathfrak{g})$.
  X is $\mathbb{Q}$-factorial $\Leftrightarrow$ All parts in the partition of $\mathbb{O}$ are equal.

For $G = Sp_n$, SO_{2n+1}, $\tilde{\mathbb{O}} \xrightarrow{\sim} \mathbb{O} \Rightarrow X$ is $\mathbb{Q}$-factorial
For $G = SO_{2n}$, X is almost always $\mathbb{Q}$-factorial.
(The exact condition can be described in terms of the partition.)

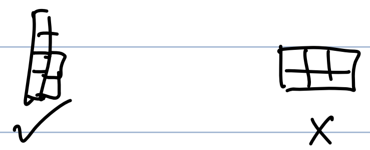
"Terminal": Defn / Propn: (Namikawa) Let Z be singular symplectic. Then, Z is terminal if $\operatorname{codim}_Z Z \setminus Z^{reg} \geq 4$.

<u>Example</u>: Let $\mathfrak{g}$ be classical, $\mathbb{O} \subset \mathfrak{g}$ be a nilpotent orbit.
Let $X = \operatorname{Spec}(\mathbb{O})$.

$$\operatorname{codim}_X X \setminus X^{reg} \geq 4 \iff \operatorname{codim}_{\overline{\mathbb{O}}} \overline{\mathbb{O}} \setminus \mathbb{O} \geq 4 \iff (*)$$

$(*) \Rightarrow$ The partition λ corresponding to $\mathbb{O}$ satisfies

$$\lambda_i \leq \lambda_{i+1} + 1.$$



<u>Theorem</u>: (Special case of BCHM) Maximal partial resolution of X exists and is $\mathbb{Q}$-factorial and terminal. Also, any $\mathbb{Q}$-factorial and terminal partial Poisson resolution is normal.

The point is that many questions about X can be addressed by studying Y.

For example, $\mathfrak{h}_X = H^2(Y^{reg}, \mathbb{C})$.

Next, we describe Y when $X = \mathbb{C}[\tilde{\mathbb{O}}]$.
The idea is to use parabolic / Lusztig - Spaltenstein induction.

Fix $(L, \tilde{\mathbb{O}}_L)$, $L \subseteq G$ Levi, $\tilde{\mathbb{O}}_L$ is an L-equivariant cover of a nilpotent orbit in $\mathfrak{l}^*$. Pick parabolic P.

(L is only reductive, and not semisimple. So, we might need to replace it by its semisimple quotient.)

$P = L \ltimes U$, $P, \tilde{\mathbb{O}}_L \rightsquigarrow$ singular symplectic variety Y with a Hamiltonian G-action.

(If $\mathbb{O}_L = \{0\}$, $Y = T^*(G/P)$, $\mathfrak{n} = \mathrm{Lie}(U)$)

$T^*G \simeq G \times \mathfrak{g}^*$ using left invariant vector fields

$G \times G \curvearrowright T^*G$ - Hamiltonian action given by:

$$(g_1, g_2) \cdot (g, \alpha) = (g_1 g g_2^{-1}, g_1 \alpha)$$

Action on the right has moment map

$$(g, \alpha) \longmapsto -\alpha \quad \leftarrow \text{To be corrected}$$

Action on the left has moment map

$$(g, \alpha) \longmapsto g \cdot \alpha$$

$X = \mathrm{Spec}(\mathbb{C}[\tilde{\mathbb{O}}_L]) \curvearrowleft L$ is Hamiltonian with moment map $\mu_L : X_L \twoheadrightarrow \overline{\mathbb{O}}_L \hookrightarrow \mathfrak{l}^*$ - finite morphism

$P \twoheadrightarrow L \curvearrowright X_L \rightsquigarrow$ Hamiltonian action of P on X_L

with moment map $\mu_L : X_L \longrightarrow \mathfrak{l}^* \oplus \mathfrak{n}^* = \mathfrak{p}^*$

Then, $T^*G \times X_L$ has a Hamiltonian P-action:

$$p \cdot (g, \alpha, x) = (gp^{-1}, \alpha, px) \text{ with moment}$$

map $\mu : (g, \alpha, x) \longmapsto -\alpha|_{\mathfrak{p}} + \mu_L(x)$

$Y := \mu^{-1}(0)/P$

$$\begin{array}{ccc} \mu^{-1}(0) = \{(g, \alpha, x) : \alpha|_{\mathfrak{p}} = \mu_L(x)\} & & (g, \alpha, x) \\ \downarrow \wr & & \downarrow \\ G \times X_L \times (\mathfrak{g}/\mathfrak{p})^* & & (g, x, \alpha - \mu_L(x)) \end{array}$$

Then, $Y = \mu^{-1}(0)/P = G \times^P (X_L \times (\mathfrak{g}/\mathfrak{p})^*)$

Given $x \in X_L$, $\beta \in (\mathfrak{g}/\mathfrak{p})^* \rightsquigarrow [g, x, \beta]$

The P-orbit of (g, x, β).

Y has a canonical Poisson structure and a Hamiltonian G-action.

<u>Important exercise</u>: $A \longrightarrow$ Poisson algebra,

$G \to$ algebraic group with $G \circlearrowright A$ rationally via Poisson algebra automorphisms.

Let $\phi : \mathfrak{g} \longrightarrow A$ be the co-moment map that is a G-

equivariant linear map s.t. $\{\phi(\xi), \cdot\} = \xi_A$ for all $\xi \in \mathfrak{g}$. Then, $(A/A\phi(\mathfrak{g}))^G$ is Poisson with bracket:

$$\{a + A\phi(\mathfrak{g}),\ b + A\phi(\mathfrak{g})\} := \{a,b\} + A\phi(\mathfrak{g})$$

This is called the Hamiltonian reduction of A.

As Y is not affine, to define $\{\cdot,\cdot\}$ on $\mathcal{O}_Y$, we need to sheafify.

Let $\omega : G \to G/P$, $\eta : Y = G \times^P (X_L \times (\mathfrak{g}/\mathfrak{p})^*) \to G/P$ be projections, which are both affine morphisms.

Let $\mathcal{U} \subseteq G/P$ be affine $\rightsquigarrow$ Can consider $\mathbb{C}[\eta^{-1}(\mathcal{U})]$.

Exercise: We can identify $\mathbb{C}[\eta^{-1}(\mathcal{U})]$ with the Hamiltonian reduction of $\mathbb{C}[\omega^{-1}(\mathcal{U}) \times \mathfrak{g}^* \times X_L]$ under the P-action.

Hence, we get a bracket $\{\cdot,\cdot\}$ on $\mathcal{O}_Y$.

Also, $G \curvearrowright Y$ whose moment map μ' we compute:

Now, $G \times P \curvearrowright T^*G \times X_L$

$\rightsquigarrow G \times P \curvearrowright \mu^{-1}(0)$

$\rightsquigarrow G \curvearrowright Y = \mu^{-1}(0)/P$

Then, moment map $G \curvearrowright Y$: $[g', x, \beta] \mapsto g'(\mu_L(x) + \beta)$

Exercise : • μ' is proper.
• μ' is indeed a moment map.
• $\mathrm{Im}(\mu')$ is the closure of a single orbit.

Now, recall that our goal was to construct a $\mathbb{Q}$-factorial terminalization.

Theorem : There is a bijection between :
1) $\tilde{\mathbb{O}}$ G-equivariant covers of nilpotent orbits in $\mathfrak{g}^*$
2) $(L, \tilde{\mathbb{O}}_L)$ L is Levi, $\tilde{\mathbb{O}}_L$ is an L-equivariant cover of a nilpotent orbit $\mathfrak{l}^*$ s.t. $X_L = \mathrm{Spec}(\mathbb{C}[\tilde{\mathbb{O}}_L])$ is a $\mathbb{Q}$-factorial terminalization.

The bijection $2 \Rightarrow 1$ is constructed as follows :
Choose P parabolic $\rightsquigarrow$ $Y = G \times^P (X_L \times (\mathfrak{g}/\mathfrak{p})^*)$ has a unique open orbit $\tilde{\mathbb{O}}$ (depending on L, $\mathbb{O}_L$, not on P). Then, Y is a $\mathbb{Q}$-factorial terminalization of $X = \mathrm{Spec}\, \mathbb{C}[\tilde{\mathbb{O}}]$.

Remark : $(L, \tilde{\mathbb{O}}_L)$ is considered upto G-conjugacy.

I) X_L is $\mathbb{Q}$-factorial and terminal $\Rightarrow$ So is Y.
II) $\mathfrak{h}_X = H^2(Y^{reg}, \mathbb{C}) \xleftarrow{\sim} (\mathfrak{l}/[\mathfrak{l}, \mathfrak{l}])^* = H^2(G/P, \mathbb{C})$
η^* is an isomorphism because $H^i(X_L^{reg}, \mathbb{C}) = 0$ for $i = 1, 2$.

For $i=1$, this is automatic.

For $i=2$, X_L is $\mathbb{Q}$-factorial, $H^0(X_L^{reg}, \mathbb{C}) = H^2(\tilde{\mathbb{O}}_L, \mathbb{C})$

III) The claim that G has an open orbit in Y is classical: It follows from

$$\dim Y = \dim \mu^{-1}(Y) \quad \text{(To be explained)}$$

IV) To show that $2 \Rightarrow 1$ is a bijection (i.e., to recover $(L, \tilde{\mathbb{O}}_L)$ from X), we use deformations. $\mathbb{C}[\tilde{\mathbb{O}}]$ admits a Poisson deformation over $\mathfrak{h}_X$ s.t. each fiber has a Hamiltonian action of G. We take the generic fiber, we look at the moment map image, which is the closure of a single orbit.

L = centralizer of semisimple part, $\mathbb{O}_L$ = L-orbit of nilpotent part.

<u>Lemma</u>: $\dim(Y) = \dim(\mu^{-1}(Y))$.

<u>Proof</u>: Deform! $\mathfrak{z} := (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^* \subset \mathfrak{p}^*$

$\rightsquigarrow Y_{\mathfrak{z}} = \mu^{-1}(\mathfrak{z})/P$

$$Y = G \times^P (X_L \times (\mathfrak{g}/\mathfrak{p})^*), \quad Y_{\mathfrak{z}} = G \times^P (X_L \times (\mathfrak{g}/\mathfrak{p})^* \times \mathfrak{z})$$

$$\begin{array}{c} Y_{\mathfrak{z}} \\ \downarrow \\ \mathfrak{z} \end{array} \leftarrow \text{equidimensional fibers}$$

Pick $z \in \mathfrak{z}$ generic $\iff G_z = L$.

<u>Facts</u>: • We have an isomorphism

$$Y_z = G \times^P (X_L \times (\mathfrak{g}/\mathfrak{p})^* \times \{z\}) \xrightarrow{\sim} G \times^L (X_L \times \{z\}).$$

• $\mu' : Y_z \longrightarrow \mathfrak{g}^* \qquad (g, x) \longmapsto g(z + \mu_L(x))$

is finite, and so, the image is the closure of a single orbit.

So, $\dim(Y) = \dim(Y_z) = \dim(\mu'(Y_z))$

Hence, it remains to show that dim of $\mu'(Y)$ $= \dim$ of $\mu'(Y_z)$.

$$\begin{array}{c} Y_{\mathbb{C}z} = \mu^{-1}(\mathbb{C}z)/P \longrightarrow \mathbb{C}z \\ \downarrow \mu' \\ \mathfrak{g}^* \\ \downarrow \pi_G \\ \mathfrak{g}^*/\!/G \end{array} \quad (= \operatorname{Spec}(\mathbb{C}[\mathfrak{g}^*]^G))$$

Then, $\operatorname{im}(\pi_G \circ \mu') = \pi_G(\mu'(Y_{\mathbb{C}z}))$, but

$\mu'(Y_{rz}) \xrightarrow[\pi_G]{}$ single point for any $r \in \mathbb{C}$.

Thus, $\mu'(Y_{rz})$ is the closure of a single orbit.

$\pi_G(z) \neq 0 \Rightarrow \pi_G(\mu'(Y_{\mathbb{C}z}))$ is a curve.

Now, $\operatorname{im}(\mu') =$ preimage of 0 under

$$\underbrace{\mu'(Y_{\mathbb{C}^2})}_{\text{irreducible}} \xrightarrow[\pi_G]{} \text{curve } \pi_G(\mu'(Y_{\mathbb{C}^2})).$$

$$\Rightarrow \dim(\operatorname{Im}\mu') = \dim(\mu'(Y_{\mathbb{C}^2})) - 1 = \dim Y.$$

Corrections:

1) $T^*G \simeq G \times \mathfrak{g}^*$ via <u>left</u> invariant vector fields.

$$(g_1, g_2)\cdot(g, \alpha) = (g_1 g g_2^{-1}, g_2 \alpha).$$

2) $T^*(G/P)$ is <u>not</u> the only possible symplectic resolution for $\operatorname{Spec}(\mathbb{C}[\tilde{\mathbb{O}}])$.

For example, $\mathbb{C}^{2n} = \operatorname{Spec}(\mathbb{C}[\tilde{\mathbb{O}}])$ for

$$\tilde{\mathbb{O}} = \mathbb{C}^{2n} \setminus \{0\} \xrightarrow[\pm 1]{} \mathbb{O}$$

$\hookrightarrow$ rank one orbit in $\mathfrak{sp}_{2n}$

(Almost) All possible symplectic resolutions are of the form

$$G \times^P ((\mathfrak{g}/\mathfrak{p})^* \times \mathbb{C}^{2n})$$

Now, let X be conical, singular symplectic.

$$\rightsquigarrow \mathfrak{h}_X, W_X$$

$$\mathfrak{h}_X = H^2(Y^{reg}, \mathbb{C})$$

where $Y \longrightarrow X$ is a $\mathbb{Q}$-factorial terminalization.

How to recover $\mathfrak{h}_X, W_X$ from X itself?

<u>Example</u>: $\Gamma \subseteq SL_2(\mathbb{C})$ be a finite subgroup

$\rightsquigarrow X = \mathbb{C}^2/\Gamma$

Upto SL_2-conjugacy, Γ's are classified by type ADE Dynkin diagrams.

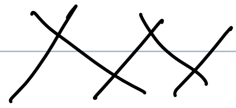
$$A_l \rightsquigarrow \Gamma = \left\{ \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} : \varepsilon^{l+1} = 1 \right\}$$

Minimal resolution of $\mathbb{C}^2/\Gamma$

Isomorphism away from zero $\leftarrow \ \uparrow \pi$

$\widetilde{\mathbb{C}^2/\Gamma}$

Then, $\pi^{-1}(\{0\}) = \bigcup \mathbb{P}^1$'s, whose intersection is either ϕ or is transversal at a single point.



$\rightsquigarrow$ Can construct a graph whose vertices are the $\mathbb{P}^1$'s and we have an edge between a pair of vertices if the corresponding $\mathbb{P}^1$'s intersect.

Then, this graph gives us the corresponding Dynkin diagram.

$Y = \widetilde{\mathbb{C}^2/\Gamma}$ is homotopy equivalent to $\pi^{-1}(0)$.

Now, $H^2(\pi^{-1}(0), \mathbb{C})$ = vector space with basis indexed by components

$\downarrow S$

$\mathfrak{h}_\Gamma$ = Cartan space of the Dynkin diagram

Basis element corresponding to $\mathbb{P}^1 \longmapsto$ simple roots

$W_\Gamma \longrightarrow$ Weyl group of the ADE Dynkin diagram

Then,

$$\mathfrak{h}_X = \mathfrak{h}_\Gamma \quad \text{and} \quad W_X = W_\Gamma .$$

In general, $\mathfrak{h}_X = \bigoplus_{i=0}^{k} \mathfrak{h}_i \curvearrowleft W_X = \prod_{i=1}^{k} W_i$

$\mathfrak{h}_0 = H^2(X^{reg}, \mathbb{C})$, $(\mathfrak{h}_i, W_i) \leftrightsquigarrow$ codimension 2 symplectic leaves

Def^n: $X \longrightarrow$ Poisson variety. An algebraic symplectic leaf in X is a locally closed subvariety $\mathcal{L} \subseteq X$ s.t.

i) $\mathcal{L}$ is irreducible, smooth.

ii) $\mathcal{L}$ is a Poisson subvariety, i.e., $\forall$ open affine $\mathcal{U} \subseteq X$, the ideal of zeroes of $\mathcal{L} \cap \mathcal{U} \subseteq \mathcal{U}$ is stable under $\{\mathbb{C}[\mathcal{U}], \cdot\}$ ($\Rightarrow \{\cdot, \cdot\}$ on $\mathcal{O}_\mathcal{L}$)

iii) $\{\cdot, \cdot\}$ on $\mathcal{L}$ is symplectic.

Example: If G is a connected algebraic group, then, the symplectic leaves in $\mathfrak{g}^* = G$-orbits.

Theorem: (Kaledin) If X is singular symplectic, then, $X = \coprod$ finitely many symplectic leaves.

Let $\mathcal{L}_1, \ldots, \mathcal{L}_k$ be codimension 2 symplectic leaves of X.

Goal : $\mathcal{L}_i \rightsquigarrow \mathfrak{h}_i, W_i$

Step 1 : Use $\mathcal{L}_i$ to construct a Kleinian singularity Γ_i.
To get Γ_i, consider (formal) transverse slice Σ_i to $\mathcal{L}_i$ in X. Then, Σ_i is a codim. 2 symplectic singularity. Then, Σ_i is a formal neighbourhood of 0 in $\mathbb{C}^2/\Gamma_i$ for a unique Γ_i.

$$\Gamma_i \rightsquigarrow (\mathfrak{h}_{\Gamma_i}, W_{\Gamma_i}).$$

It turns out that $\pi_1(\mathcal{L}_i) \curvearrowright \mathfrak{h}_{\Gamma_i}, W_{\Gamma_i}$

Then, $\mathfrak{h}_i := \mathfrak{h}_{\Gamma_i}^{\pi_1(\mathcal{L}_i)}$, $W_i := W_i^{\pi_1(\mathcal{L}_i)}$

Construction of the $\pi_1(\mathcal{L}_i)$-action : (monodromy)

$\mathbb{Q}$-factorial terminalization

$$\begin{tikzcd} Y \arrow[r, "\pi_Y"] & X \\ Y \times_X \Sigma_i \arrow[u, hook] \arrow[r] & \Sigma_i \arrow[u, hook] \end{tikzcd}$$

symplectic resolution because Y is terminal

$\Rightarrow Y \times_X \Sigma_i$ is (completion at $\pi^{-1}(0)$) of $\widetilde{\mathbb{C}^2/\Gamma_i}$. So,
$\forall\, y \in \mathcal{L}_i$, $\pi_Y^{-1}(y) \simeq \pi^{-1}(0) \hookleftarrow \mathbb{C}^2/\Gamma_i \longleftarrow \widetilde{\mathbb{C}^2/\Gamma_i} : \pi$

Varying $y \rightsquigarrow \pi_1(\mathcal{L}_i, y) \curvearrowright$ components on $\pi_Y^{-1}(y)$.
$\rightsquigarrow$ Action of $\pi_1(\mathcal{L}_i)$ on $\mathfrak{h}_{\Gamma_i}, W_{\Gamma_i}$ by diagram automorphisms.

Then, we can define

$$\mathfrak{h}_i = \mathfrak{h}_{\Gamma_i}^{\pi_1(\mathcal{L}_i)}, \quad W_i = W_{\Gamma_i}^{\pi_1(\mathcal{L}_i)}.$$

Example: (Reference: Slodowy's book)

$\mathfrak{g} \longrightarrow$ simple Lie algebra, $\mathcal{N} \subseteq \mathfrak{g}$

Leaves in $\mathcal{N}$ = orbits.

Fact: $\exists!$ codim 2 leaves, a.k.a., semiregular orbit.

For $\mathfrak{sl}_n$, we have $(n-1, 1)$ (reg. $= (n)$)

" $\mathfrak{so}_{2n+1}$, " " $(2n-1, 1^2)$ (reg. $= (2n+1)$)

" $\mathfrak{sp}_{2n}$, " " $(2n-2, 2)$ (reg. $= (2n)$)

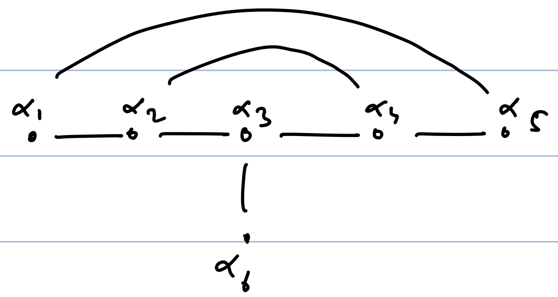
" $\mathfrak{so}_{2n}$, " " $(2n-3, 3)$ (reg. $= (2n-1, 1)$)

$$H^2(\underset{\substack{\shortparallel \\ \mathcal{O}^{reg}}}{X^{reg}}, \mathbb{C}) = \{0\}. \text{ Hence, } \mathfrak{h}_X = \mathfrak{h}_{\perp},\ W_X = W_{\perp}.$$

If $\mathfrak{g}$ is of ADE type, then Γ is of the same type and the monodromy action is trivial.

$\mathfrak{g}$	$\Sigma_{\perp}$	$\pi_1(\mathcal{O}_{subreg.})$
B_n	A_{2n+1}	$\mathbb{Z}/2\mathbb{Z}$
C_n	D_{n-1}	$\mathbb{Z}/2\mathbb{Z}$
F_4	E_6	$\mathbb{Z}/2\mathbb{Z}$
G_2	D_4	S_3

Then, $\pi_1(\mathcal{L}_i)$ acts on the simply laced Dynkin diagram by folding. For example, for E_6,



$\Rightarrow \alpha_1, \alpha_3, \alpha_2+\alpha_4, \alpha_1+\alpha_5$ which is the root system of F_4.

$H^2(\cdot, \mathbb{C}) \rightsquigarrow \mathrm{Pic}(\cdot) \otimes_{\mathbb{Z}} \mathbb{C}$

$\mathrm{Pic}(Y^{reg}) = [\, Y^{reg} = X^{reg} \amalg$ divisors in the pre-image of co-dim 2 leaves $\amalg$ rest $]$

The "rest" has codim >1, and so, doesn't affect the Picard group.

$$\Rightarrow \mathrm{Pic}(Y^{reg}) \otimes_{\mathbb{Z}} \mathbb{C} = \mathrm{Pic}(X^{reg}) \otimes_{\mathbb{Z}} \mathbb{C} \leftarrow h_0$$

$$\begin{array}{ccc} \| & & \oplus \\ H^2(Y^{reg}, \mathbb{C}) & & \bigoplus_{\text{irred. comp.}} \mathbb{C} = \bigoplus_{i=1}^{k} h_i \end{array}$$

16/06/22

Recap: $G \to$ semisimple $\quad \tilde{\mathbb{O}} \to$ G-equivariant cover of a nilpotent orbit in $\mathfrak{g}^*$.

$\rightsquigarrow A = \mathbb{C}[\tilde{\mathbb{O}}]$ is graded Poisson.

$\rightsquigarrow X = \mathrm{Spec}(A)$ is conical singular symplectic

Goal: Construct filtered quantizations of A.

Upto isomorphism, they are parametrized by $\mathfrak{h}_X/W_X$.

For this, we construct $Y \to X$, a $\mathbb{Q}$-factorial terminalization and take $\mathfrak{h}_X = H^2(Y^{reg}, \mathbb{C})$.

Construction of Y: Take Levi $L \subseteq G$, L-equivariant nilpotent cover $\tilde{\mathbb{O}}_L$ and $X_L = \operatorname{Spec}(\mathbb{C}(\tilde{\mathbb{O}}_L))$ is $\mathbb{Q}$-factorial terminal.

Choose parabolic P with Levi L

$\rightsquigarrow$ $Y = G \times^P (X_L \times (\mathfrak{g}/\mathfrak{p})^*)$ as Hamiltonian reduction.

$$P \curvearrowright T^*G \times X_L = G \times \mathfrak{g}^* \times X_L \quad p\cdot(g,\alpha,x) = (gp^{-1}, p\alpha, px)$$

$$\mu : G \times \mathfrak{g}^* \times X_L \longrightarrow \mathfrak{p}^* \quad (g,\alpha,x) \longmapsto -\alpha_{\mathfrak{p}} + \mu_L(x)$$

$$Y = \mu^{-1}(0)/P \rightsquigarrow \{\cdot,\cdot\} \text{ on } \mathcal{O}_Y.$$

Finally, take $\mathfrak{h}_X = H^2(Y^{reg}, \mathbb{C}) \xleftarrow{\sim} (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$

Then, $\tilde{\mathbb{O}}$ is the open G-orbit in Y (this only depends on L, $\tilde{\mathbb{O}}_L$, and not P) $\Rightarrow$ $Y \to X$ is a $\mathbb{Q}$-factorial terminalization.

$\mathbb{C}^* \curvearrowright X_L$, $\deg(\{\cdot,\cdot\}) = -d$

$$\Rightarrow \mathbb{C}^* \curvearrowright Y \quad z\cdot(g,\alpha,x) = (g, z^{-d}\alpha, z\cdot x)$$

rescales $\{\cdot,\cdot\}$ on $\mathcal{O}_Y$ by $z \longrightarrow z^{-d}$.

Quantization of Y

$$\eta : Y = G\times^P (X_L, (\mathfrak{g}/\mathfrak{p})^*) \longrightarrow G/P$$

is $\mathbb{C}^*$-invariant and affine.

$\rightsquigarrow \eta_* \mathcal{O}_Y$ a sheaf of graded Poisson algebras on G/P.

Def^n: A filtered quantization of Y is a quasi-coherent sheaf $\mathcal{D}$ on G/P of $\mathbb{C}$-algebras with algebra filtrations given by sheaves of $\mathcal{O}_{G/P}$-modules $\mathcal{D} = \bigcup_{i \geq 0} \mathcal{D}_i$ s.t. $[\mathcal{D}_{\leq i}, \mathcal{D}_{\leq j}] \subseteq \mathcal{D}_{i+j-d}$ ($\rightsquigarrow$ $\mathrm{gr}\,\mathcal{D}$ is a sheaf of graded Poisson algebras) and $\mathrm{gr}\,\mathcal{D} \xrightarrow{\sim} \eta_* \mathcal{O}_Y$ of sheaves of graded Poisson algebras.

Example: $Y = T^*(G/P)$, $\mathcal{D} := \mathcal{D}_{G/P}$ - sheaf of linear algebraic differential operators on G/P filtered by order of differential operator is a filtered quantization of $T^*(G/P)$. $(d=1)$.

Temporary goal: For $\lambda \in (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$, produce a filtered quantization $\mathcal{D}_\lambda$ of Y (by quantizing the construction of Y).

Quantum analog of $T^*G \times X_L$:

Recall that $H^2(X_L^{reg}, \mathbb{C}) = \{0\}$
$\Rightarrow$ $\exists!$ filtered quantization $\mathcal{A}_L$ of $\mathbb{C}[X_L]$.
$\mathcal{D}(G)$ (the algebra of differential operators on G)

is a filtered quantization of $\mathbb{C}[T^*G]$.
$\rightsquigarrow \mathcal{D}(G) \otimes \mathcal{A}_L$ is a filtered quantization of $T^*G \times X_L$.

Classical comoment map $\phi : \mathfrak{p} \longrightarrow \mathbb{C}[T^*G \times X_L]$.
Then, $\phi(\xi) = -\underbrace{\xi_r}_{\text{left-invariant}} \otimes 1 + 1 \otimes \phi_L(\xi) \leftarrow$ degree d.

Def^n: $\mathcal{A} \rightarrow$ associative algebra, $R \rightarrow$ algebraic group acting rationally on R by algebra automorphisms. A quantum co-moment map is an R-equivariant linear map $\Phi : \mathfrak{r} \, (= \mathrm{Lie}(R)) \longrightarrow \mathcal{A}$ s.t.

$$[\Phi(\xi), \cdot\,] = \xi_{\mathcal{A}} \quad \forall\, \xi \in \mathfrak{r}.$$

Example: $G \curvearrowright G$ by right translations $\rightsquigarrow G \curvearrowright \mathcal{D}(G)$
Then, $\Phi : \xi \longmapsto -\xi_r$ (for left translations $\xi \mapsto \xi_l$)
is the quantum comoment map.

Exercise: $\phi_L : \mathfrak{l} \longrightarrow \mathbb{C}[X_L]_d$ comoment map
(works for all $\tilde{O}_L$). Then, $\exists$ Lie algebra homo.
$$\Phi_L : \mathfrak{l} \longrightarrow (\mathcal{A}_L)_{\leq d} \quad \text{s.t.} \quad \phi_L = \Phi_L + (\mathcal{A}_L)_{\leq d-1}$$
Such a Φ_L is unique if $\Phi_L|_{\mathfrak{z}(\mathfrak{l})} = 0$.

Finally, $L \curvearrowright \mathcal{A}_L$ by filtered algebra automorphisms s.t. Φ_L is the quantum comoment map.

We can view Φ_L as $\mathfrak{p} \twoheadrightarrow \mathfrak{l} \longrightarrow (\mathcal{A}_L)_{\le d}$.

Pick $\lambda \in (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ and $\rho_{G/P} = \frac{1}{2}\left(\begin{array}{c}\text{char of action of } \mathfrak{l} \\ \text{on } \Lambda^{top}(G/P)\end{array}\right)$

Define $\Phi_\lambda(\xi) = -\xi_r \otimes 1 + 1 \otimes \Phi_L(\xi) - \langle \lambda + \rho_{G/P}, \xi \rangle$

which is a map $\mathfrak{p} \longrightarrow (\mathcal{D}(G) \otimes \mathcal{A}_L)_{\le d}$.

The top degree part of Φ_λ is ϕ.

Exercise: Keeping the notation $\mathcal{A}, R, \Phi$. The space $(\mathcal{A}/\mathcal{A}\,\Phi(\mathfrak{r}))^R$ is an associative algebra w.r.t.

$$(a + \mathcal{A}\,\Phi(\mathfrak{r})) \cdot (b + \mathcal{A}\,\Phi(\mathfrak{r})) := ab + \mathcal{A}\,\phi(\mathfrak{r}).$$

This is called the quantum Hamiltonian reduction.

Remark: If $\mathcal{A}$ is filtered with degree $[\cdot,\cdot] \le -d$, $\operatorname{Im}(\Phi) \subseteq \mathcal{A}_{\le d}$, then $[\mathcal{A}/\mathcal{A}\,\Phi(\mathfrak{r})]^R$ inherits a filtration with degree $[\cdot,\cdot] \le -\alpha$.

We view $\mathcal{D}(G) \otimes \mathcal{A}_L$ as a quasi-coherent sheaf on G/P via $\omega: G \longrightarrow G/P$.

$\rightsquigarrow \mathcal{D}_\lambda := [(\mathcal{D}(G) \otimes \mathcal{A}_L)/(\mathcal{D}(G) \otimes \mathcal{A}_L)\,\phi_\lambda(\mathfrak{p})]^P$

— sheaf of filtered algebras on G/P.

Fact: $\mathcal{D}_\lambda$ is a filtered quantization of $\eta_* \mathcal{O}_Y$.

(Thanks to $P \curvearrowright \mu^{-1}(0)$ is free.)

Example : $X = \mathcal{N}$, $Y = T^*(G/B)$

$\rho_{G/B} = \rho = \frac{1}{2} \sum$ positive roots

- $\mathcal{D}_\lambda = \mathcal{D}_{G/B}^{\lambda-\rho}$ $(\lambda-\rho)$-twisted differential operators
- Directly generalizes to $Y = T^*(G/P)$.

Remark : $\lambda \longmapsto \mathcal{D}_\lambda$ gives a bijection

$$\begin{tikzcd} (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^* \arrow[r, "\sim"] & \{\text{filtered quantization of } Y\}/\text{iso} \arrow[dl, "\sim"', "\text{Period map (Bezrukavnikov - Kaledin)}"] \\ H^2(Y^{reg}, \mathbb{C}) \arrow[u, "\wr"] & \end{tikzcd}$$

Then, Period $(\mathcal{D}_\lambda) = \lambda$.

Quantizations of $A = \mathbb{C}[X]$

Observe that $Y \xrightarrow{\pi} X$

proper, birational   normal

$$\Rightarrow \quad \mathbb{C}[X] \xrightarrow[\sim]{\pi^*} \mathbb{C}[Y]$$

Proposition : $\mathcal{A}_\lambda := \Gamma(\mathcal{D}_\lambda)$ is a quantization of $\mathbb{C}[X]$.

Sketch of proof : Need to check that

$$\begin{tikzcd} gr(\mathcal{A}_\lambda) = gr(\Gamma(\mathcal{D}_\lambda)) \arrow[r, "\sim"] \arrow[d] & \mathbb{C}[X] \arrow[d, phantom, "\shortparallel"] \\ \Gamma(gr(\mathcal{D}_\lambda)) & \Gamma(\mathcal{O}_Y) \end{tikzcd}$$

$$\| \\ \Gamma(\mathrm{gr}(\mathcal{D}_\lambda))$$

Fact : $H^i(G/P, \eta_* \mathcal{O}_Y) = H^i(Y, \mathcal{O}_Y) = 0 \quad \forall\ i > 0$
This follows from X having rational singularities.
(Beauville).

Hence, to show that $\mathrm{gr}(\Gamma(\mathcal{D}_\lambda)) = \Gamma(\mathrm{gr}(\mathcal{D}_\lambda))$,
we'll use that $H^i(G/P, \eta_* \mathcal{O}_Y) = 0$

$$\| \\ \bigoplus_{i \geq 0} (\eta_* \mathcal{O}_Y)_i$$

$\mathrm{gr}\, \mathcal{D}_\lambda \simeq \eta_* \mathcal{O}_Y \longleftrightarrow$ We have a SES $\forall\ i$

$$0 \longrightarrow (\mathcal{D}_\lambda)_{\leq i-1} \longrightarrow (\mathcal{D}_\lambda)_{\leq i} \longrightarrow (\eta_* \mathcal{O}_Y) \longrightarrow 0$$

Has $H^1 = 0$

$\Rightarrow H^1((\mathcal{D}_\lambda)_{\leq i-1}) = 0 \ \forall\ i$

$\Rightarrow$ We have a SES

$$0 \longrightarrow \Gamma((\mathcal{D}_\lambda)_{\leq i-1}) \longrightarrow \Gamma((\mathcal{D}_\lambda)_{\leq i}) \longrightarrow \Gamma(\eta_* \mathcal{O}_Y) \longrightarrow 0$$

$\Leftrightarrow \mathrm{gr}(\Gamma(\mathcal{D}_X)) = \Gamma(\mathrm{gr}(\mathcal{D}_X))$.

Example : $X = \mathcal{N}$, $Y = T^*(G/B)$, $\mathcal{D}_\lambda = \mathcal{D}^{\lambda - \rho}_{G/B}$.
Then, $\Gamma(\mathcal{D}^{\lambda-\rho}_{G/B}) = \mathcal{U}_\lambda \ (= \mathcal{U}\mathfrak{g} / \mathcal{U}\mathfrak{g}\, m_\lambda)$, which is
(By B.B. localization) a quantization of $\mathbb{C}[X]$

Fact : $\lambda \longmapsto \mathcal{A}_\lambda$ gives the bijection
$$\mathfrak{h}_X / W_X \xrightarrow{\sim} \{\text{quantizations of } \mathbb{C}[X]\}/\text{iso.},$$
which is exactly the classification theorem we stated earlier.

Remarks : 1) When do we have $\mathcal{A}_\lambda \simeq \mathcal{A}_{\lambda'}$ as filtered algebras with G-action?

$N_G(L) \subseteq G$, $N_G(L) \curvearrowright \{L\text{-equivariant nilpotent covers}\}$

By twisting L-action (& moment map to $\mathfrak{l}^*$).

$\rightsquigarrow N_G(L, \tilde{\mathbb{O}}_L) \subset N_G(L)$ - stabilizer of $\tilde{\mathbb{O}}_L$.

$L^\triangledown \rightsquigarrow N_G(L, \tilde{\mathbb{O}}_L)/L \curvearrowright (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$

Claim : $\mathcal{A}_\lambda \xrightarrow{\sim} \mathcal{A}_{\lambda'}$ as filtered algebras with G-action

$\Updownarrow$

λ and $\lambda' \in$ same $\underbrace{N_G(L, \tilde{\mathbb{O}}_L)/L}_{\text{finite group}}$-orbit

Hint : Both Y, $\mathcal{D}_\lambda$ depend on choice of P.
$\rightsquigarrow Y^P, \mathcal{D}_\lambda^P$. Turns out that $\mathcal{A}_\lambda$ doesn't depend on P, as a filtered quantization.

$n \in N_G(L, \tilde{\mathbb{O}}_L) \rightsquigarrow nPn^{-1}$ another Parabolic with Levi L.

$$Y^{nPn^{-1}} \underset{n}{\xleftarrow{\sim}} Y, \quad \mathcal{D}_{n\lambda}^{nPn^{-1}} \underset{n}{\xleftarrow{\sim}} \mathcal{D}_\lambda^P$$

$$\Rightarrow \mathcal{A}_\lambda = \Gamma(\mathcal{D}_\lambda^P) \xrightarrow{\sim} \Gamma(\mathcal{D}_{n\lambda}^{nPn^{-1}}) = \mathcal{A}_{n\lambda}.$$

In the opposite direction, let $\Psi : \mathcal{A}_\lambda \xrightarrow{\sim} \mathcal{A}_{\lambda'}$ be a filtered algebra isomorphism that is G-equivariant.

$\rightsquigarrow \mathrm{gr}\,\Psi \in \mathrm{Aut}_G\, \mathbb{C}[X]$ - automorphisms of graded Poisson algebras, G-equiv

$\rightsquigarrow$ group homomorphism

$$N_G(L, \tilde{\mathbb{O}}_L)/L \longrightarrow \mathrm{Aut}_G(\mathbb{C}[X]).$$

<u>Fact</u>: (Losev, Namikawa) There is a short exact sequence:

$$L \longrightarrow W_X \longrightarrow N_G(L, \tilde{\mathbb{O}}_L)/L \longrightarrow \mathrm{Aut}_G(\mathbb{C}[X]) \longrightarrow 1$$

$$\begin{array}{ccc} \cup & & \\ n & \longmapsto & \mathrm{gr}[\mathcal{A}_\lambda \xrightarrow{\sim} \mathcal{A}_{n\lambda}]. \end{array}$$

Using this construction and classical analog, we get an algebraic orbit method. (conjectured by Vogan.)

<u>Theorem</u>: (LMBM '21) $G \longrightarrow$ simply connected. Then, $\exists$ a bijection between:

1) Filtered quantizations of $\mathbb{C}[\tilde{\mathbb{O}}]$ $\forall$ nilpotent covers $\tilde{\mathbb{O}}$ upto filtered algebra isomorphisms.
(NOT upto filtered quantization isomorphisms.)

2) All G-equivariant covers of all (co)-adjoint orbits.

In the above bijection,

Nilp $\tilde{\mathbb{O}}$ in 2) $\longmapsto$ its quantization $\mathcal{A}_0$
(the canonical quantization)

Can the algebras $\mathcal{A}_\lambda$ be described explicitly?

$G \curvearrowright \mathcal{A}_\lambda$ with quantum co-moment map

$$\Phi : \mathcal{U}\mathfrak{g} \longrightarrow \mathcal{A}_\lambda . \quad \text{(See earlier exercise)}$$

<u>Theorem</u>: (LMBM '21, MBM '21)

$\lambda = 0 \implies \ker \Phi_G$ is maximal ideal.

(We can recover ker explicitly $\forall \, \tilde{\mathbb{O}}$).

Also, if $\tilde{\mathbb{O}} \leq \mathfrak{g}^*$, then $\underline{\Phi_G : \mathcal{U}\mathfrak{g} \twoheadrightarrow \mathcal{A}_0}$.

<u>Harishchandra bimodules</u>

Def^n: (Harishchandra) An HC $\mathcal{U}\mathfrak{g}$-bimodule $\mathcal{B}$ is a f.g. $\mathcal{U}\mathfrak{g}$-bimodule s.t. the adjoint $\mathfrak{g}$-action is locally finite, i.e., $\forall \, b \in \mathcal{B}$, $\exists$ a f.d. $\mathrm{ad}(\mathfrak{g})$-stable $\mathcal{B}_0 \subset \mathcal{B}$ and $b \in \mathcal{B}_0$.

(Here, $\mathrm{ad}(\xi) b = \xi \cdot b - b \cdot \xi$ for $\xi \in \mathfrak{g}, b \in \mathcal{B}$)

<u>Example</u>: 1) $\mathcal{B} = \mathcal{U}\mathfrak{g}$, called the regular bimodule, is an HC-bimodule. This follows by observing that

$$\mathcal{U}\mathfrak{g} = \bigcup_{i \geq 0} \underbrace{(\mathcal{U}\mathfrak{g})_{\leq i}}_{\text{f.d., ad}(\mathfrak{g})\text{-stable}} \quad \text{(PBW filtration)}$$

2) All sub-quotients of HC bimodules are HC.

<u>Exercise</u>: $\forall$ f.d. $\mathfrak{g}$-rep., $V \otimes \mathcal{U}\mathfrak{g}$ is an HC-bimod.

with, $(v \otimes a)\xi = v \otimes a\xi$ $\quad v \in V,\ a \in U\mathfrak{g}$

$\xi(v \otimes a) = \xi v \otimes a + v \otimes \xi a$ $\quad \xi \in \mathfrak{g}.$

a) This bimodule is HC.

b) Every HC bimodule is a quotient of $V \otimes U\mathfrak{g}$ for some V.

$G \to$ simply connected

An irreducible unitary G-representation $\rightsquigarrow$ HC-bimodule, (Take the subspace of "algebraic" vectors) which is irreducible and "unitarizable", that is, the original unitary structure restricts to a positive definite one on the bimodule.

Theorem: (Harishchandra) This defines a bijection between:

1) Unitary G-irreps.

2) Irreducible unitarizable HC-bimodules.

Experimental evidence: Unitarizable irred. HC-bimodules have "large intersection" with HC-bimodules over quantizations of $\mathbb{C}[\tilde{\mathbb{O}}]$.

Barbasch - Vogan construction and glimpses of symplectic duality

$\Phi_G : \mathcal{U}\mathfrak{g} \longrightarrow \mathcal{A}_0 \longrightarrow$ canonical quantization of $\mathbb{C}[\tilde{\mathbb{O}}]$.

$\ker \Phi_G \rightarrow$ maximal ideal

Unipotent representations of real semisimple groups

$\updownarrow \sim$ orbit method

Nilpotent orbits (and their covers)

1985 : Barbasch, Vogan defined special unipotent representations.

$I \subseteq \mathcal{U}\mathfrak{g}$ 2-sided ideal $\rightsquigarrow I \cap Z = Z_{\mathfrak{g}}$ (= center)

$\downarrow$ max. ideal w.r.t. $\subseteq$

$\downarrow S$

$\mathbb{C}[\mathfrak{h}^*]^W$

So, $I \rightsquigarrow \mathfrak{h}^*/W$.

Fact : This defines a bijection

$$\{\text{max. ideals of } \mathcal{U}\mathfrak{g}\} \xrightarrow{\sim} \mathfrak{h}^*/W$$

Notation : For $\lambda \in \mathfrak{h}^*/W$, $I_{max}(\lambda)$ = corresponding maximal ideal

Example : $I_{max}(\rho) = \mathfrak{g} \cdot \mathcal{U}\mathfrak{g}$, $I_{max}(0) = \mathcal{U}\mathfrak{g} \cdot \mathfrak{m}_0$

BV : Collection of max. ideals ($\longleftrightarrow$ subsets in $\mathfrak{h}^*/W$).

Let $\mathfrak{g}^\vee$ be Langlands dual. ($\mathfrak{g} = \mathfrak{so}_{2n+1} \longleftrightarrow \mathfrak{g}^\vee = \mathfrak{sp}_{2n}$)

Let $\mathbb{O}^\vee \subseteq \mathfrak{g}^\vee$ be a nilpotent orbit $\rightsquigarrow (e^\vee, h^\vee, f^\vee)$.

Can conjugate h to be inside $\mathfrak{h}^\vee = \mathfrak{h}^*$.

Def^n: (BV '85) $I_{\mathbb{O}^\vee} = I_{max}(\frac{1}{2}h^\vee)$
(special unipotent ideal).

Theorem: (LMBM '21). $\forall\, \mathbb{O}^\vee$, $\exists$ an Ad(g)-equivariant cover of nilpotent orbit $\tilde{d}(\mathbb{O}^\vee)$ s.t. $I_{\mathbb{O}^\vee} = \mathrm{Ker}(\Phi_G : \mathcal{U}\mathfrak{g})$ is a canonical quantization of $\mathbb{C}[\tilde{d}(\mathbb{O}^\vee)]$.
- $\mathbb{O}^\vee \longmapsto \tilde{d}(\mathbb{O}^\vee)$ gives an embedding
$\{$nilpotent orbits in $\mathfrak{g}^\vee\} \hookrightarrow \{$nilpotent covers for $\mathfrak{g}\}$

Barbasch-Vogan duality: $I \subseteq \mathcal{U}\mathfrak{g}$ 2-sided ideal.

$\rightsquigarrow \mathrm{gr}\, I \subseteq S\mathfrak{g} = \mathbb{C}[\mathfrak{g}^*]$ for the PBW filtration

$\downarrow$ homogeneous G-stable ideal

$\rightsquigarrow \mathrm{gr}\, I \subseteq \mathfrak{g}^*$

$\quad\;\; \|$

$V(I) \longrightarrow$ associated variety

If I is maximal, $V(I)$ is the closure of a single nilpotent orbit.

Eg: If $I = \mathrm{Ker}[\Phi_G : \mathcal{U}\mathfrak{g} \longrightarrow \mathcal{A}]$,
$\hookrightarrow$ quantization of $\mathbb{C}[\tilde{\mathbb{O}}]$

then $V(I) = \bar{\mathbb{O}}$, where $\tilde{\mathbb{O}}$ is a cover of $\bar{\mathbb{O}}$.

Def^n: (BV '85) BV dual $d(\mathbb{O}^\vee)$ of $\mathbb{O}^\vee \subseteq \mathfrak{g}^\vee$ is then an orbit in $V(I_{\mathbb{O}^\vee})$.

$\rightsquigarrow d : \{\text{nilpotent orbits in } \mathfrak{g}^\vee\}$

$\longrightarrow \{\text{nilpotent orbits in } \mathfrak{g}\}$

$\operatorname{im} d =$ "special" nilpotent orbits.

Examples:

- $\mathbb{O}^\vee = \{0\} \Rightarrow h^\vee = 0 \Rightarrow I_{\mathbb{O}^\vee} = I_{max}(0) \Rightarrow V(I_{max}(0)) = \mathcal{N}$

  $\Rightarrow d(\mathbb{O}^\vee) =$ principal nilpotent orbit.
- $\mathbb{O}^\vee =$ principal nilpotent orbit in $\mathfrak{g}^\vee$.

  Then, it's possible to choose $h^\vee = 2\rho \in \mathfrak{h}^\vee$

  $\Rightarrow I_{\mathbb{O}^\vee} = I_{max}(\rho) = \mathfrak{g}\,\mathcal{U}\mathfrak{g} \Rightarrow V(I_{\mathbb{O}^\vee}) = \{0\}$.
- $\mathfrak{g} = \mathfrak{sl}_n = \mathfrak{g}^\vee$

  Given a partition $\mu \vdash n \rightsquigarrow \mathbb{O}_\mu$ nilpotent orbit with Jordan type μ.

  Then, $d(\mathbb{O}_\mu) = \mathbb{O}_{\mu^t}$.
- $\mathfrak{g} = \mathfrak{sp}_4$. Then, $\mathfrak{g}^\vee = \mathfrak{so}_5 = \mathfrak{sp}_4$

$$\begin{array}{ccc}
\mathbb{O}^\vee & & \mathbb{O} \\
\{0\} & \longrightarrow & (4) \\
(2,1,1) & \longrightarrow & (2,2) \\
(2,2) & \nearrow & (2,1,1) \longrightarrow \text{Not special} \\
(4) & \longrightarrow & \{0\}
\end{array}$$

Construction of $\tilde{d}$: $\tilde{d}(\mathbb{O}^\vee)$ should be a cover of $d(\mathbb{O}^\vee)$.

Case 1 : $\mathbb{O}^\vee \cap$ proper Levi subalgebra of $\mathfrak{g}^\vee = \phi$ (distinguished)

$\Leftrightarrow Z_G(e^\vee, h^\vee, f^\vee)$ is finite.

Then, $\tilde{d}(\mathbb{O}^\vee)$ = universal $Ad(\mathfrak{g})$-equivariant cover of $d(\mathbb{O}^\vee)$.

General case: Pick minimal Levi $\mathfrak{l}^\vee \subset \mathfrak{g}^\vee$ containing $e^\vee$.

$\longleftrightarrow \mathfrak{l} \subset \mathfrak{g}$, $\mathbb{O}_L^\vee = L^\vee e^\vee \rightsquigarrow \tilde{d}_L(\mathbb{O}_L^\vee)$.

$\rightsquigarrow X_L = \operatorname{Spec} \mathbb{C}[\tilde{d}_L(\tilde{\mathbb{O}}_L)]$, $P = L \ltimes \mathcal{U}$

$Y = G \times^P (X_L \times (\mathfrak{g}/\mathfrak{p})^*)$.

$\tilde{d}(\mathbb{O}^\vee)$ = open G-orbit in Y.

Example: $\cdot\ \mathfrak{g} = \mathfrak{sl}_n \Rightarrow \tilde{d}(\mathbb{O}^\vee) = d(\mathbb{O}^\vee)$

$\cdot\ \mathfrak{g} = \mathfrak{sp}_4$: $\tilde{d}$: $(2,1,1) \longmapsto$ orbit $(2,2)$

$(2,2) \longmapsto$ double cover of

(probably)

Symplectic Duality (Braden, Licata, Proudfoot, Webster)

of conical symplectic singularities (with some decoration.)

X vs $X^\vee$

The duality swaps some invariants.

$X = \operatorname{Spec} \mathbb{C}[\tilde{d}(\mathbb{O}^\vee)]$

$X^\vee = \mathcal{N}^\vee \cap$ Slodowy slice $\underline{e^\vee + \mathfrak{z}_{\mathfrak{g}^\vee}(f^\vee)}$

transverse to $\mathbb{O}^\vee$

Pair of invariants :

$X \longmapsto (\mathfrak{h}_X, \mathfrak{t}_X =$ Lie algebra of max. torus in $\{$graded Poisson auto. of $\mathbb{C}[X]\})$

Expectation : $\mathfrak{h}_{X^\vee} = \mathfrak{t}_X$ and $\mathfrak{t}_{X^\vee} = \mathfrak{h}_X$.

17/06/22

HC Bimodules

$A \to$ f.g. graded Poisson algebra, $A_i = \{0\}\ \forall i<0$ and $A_0 = \mathbb{C}$

$\mathcal{A} \to$ filtered quantization of A

Def^n : Let $\mathcal{B}$ be an $\mathcal{A}$-bimodule.

1) A good filtration on $\mathcal{B}$ is $\mathcal{B} = \bigcup_{j \in \mathbb{Z}} \underbrace{\mathcal{B}_{\leq j}}_{\text{vector space}}$ s.t.

   i) $\mathcal{B}_j = \{0\}$ for $j<<0$

   ii) Bimodule filtration $\mathcal{A}_{\leq i}\mathcal{B}_{\leq j}$, $\mathcal{B}_{\leq j}\mathcal{A}_{\leq i} \subseteq \mathcal{B}_{i+j}$.
   ($\Rightarrow$ gr $\mathcal{B}$ is an A-bimodule)

   iii) $[\mathcal{A}_{\leq i}, \mathcal{B}_{\leq j}] \subseteq \mathcal{B}_{i+j-d}$ ($\deg \{\cdot,\cdot\} = -d$)
   ($\Rightarrow$ left and right A-actions on $\mathcal{B}$-coincide, so gr $\mathcal{B}$ is an A-module.)

   iv) gr $\mathcal{B}$ is finitely generated over A.

2) We say $\mathcal{B}$ is HC if it admits a good filtration.

Example : 1) $\mathcal{A}$, the regular bimodule, is HC.
(with good filtration given by the quantization filtration)
2) Subquotients of HC bimodules are HC.

Remarks : 1) Good filtrations aren't unique. However, if $\mathcal{B} = \bigcup_j \mathcal{B}_{\leq j} = \bigcup_j \mathcal{B}'_{\leq j}$ are both good filtrations, then $\exists\, m_1, m_2 \in \mathbb{Z}$ s.t.
$$\mathcal{B}_{\leq j+m_1} \subseteq \mathcal{B}'_{\leq j} \subseteq \mathcal{B}_{\leq j+m_2} \quad \forall j \in \mathbb{Z}$$
(Exercise)
2) When $\mathcal{A} = \mathcal{U}\mathfrak{g}$, this def^n of HC is equivalent to the one stated earlier.

HC bimodules over quantizations of $\mathbb{C}[\tilde{O}]$ $(= A)$.

$\mathcal{A}$ - filtered quantization, quantum comoment map
$$\Phi : \mathcal{U}\mathfrak{g} \longrightarrow \mathcal{A}.$$
$\rightsquigarrow$ Every $\mathcal{A}$ - bimodule becomes a $\mathcal{U}\mathfrak{g}$ - bimodule.

Lemma : If B is an HC $\mathcal{A}$ - bimodule, it is also HC as a $\mathcal{U}\mathfrak{g}$ - bimodule.

Proof : $\Phi(\mathfrak{g}) \subseteq \mathcal{A}_{\leq d} \overset{\text{by (iii)}}{\Longrightarrow} [\Phi(\mathfrak{g}), \mathcal{B}_{\leq j}] \subseteq \mathcal{B}_{\leq j}$
Since $\operatorname{gr} \mathcal{B}$ is finitely generated over A and A is

positively graded, $(\mathrm{gr}\,\mathcal{B})_j$ are finite dimensional.
$\underset{\text{by (i)}}{\Rightarrow}$ $\mathcal{B}_{\leq j}$ are finite dimensional $\forall j$.

So, $\forall\, b \in \mathcal{B}$ can be included into f.d. $\mathrm{ad}(\mathfrak{g})$-stable subspaces.

It remains to see that $\mathcal{B}$ is f.g. over $\mathcal{U}\mathfrak{g}$.

Now, $\mathrm{gr}\,\mathcal{B}$ is f.g. over A $\Rightarrow$ $\mathcal{B}$ is f.g. as a left $\mathcal{A}$-module.

$\mathbb{C}[\tilde{\mathbb{O}}]$ is finite generated over $S(\mathfrak{g})$ $\Rightarrow$ $\mathcal{A}$ is finitely generated left module over $\mathcal{U}\mathfrak{g}$ $\Rightarrow$ $\mathcal{B}$ is finitely generated over $\mathcal{U}\mathfrak{g}$.

Classification, application and generalization

Classification result: Let X be conical singular symplectic. $A = \mathbb{C}[X]$. $\mathcal{A} \to$ filtered quantization

$\rightsquigarrow$ $HC(\mathcal{A})$ = full subcategory of $\mathrm{Bimod}(\mathcal{A})$ whose objects are HC.

Remark: Every f.d. bimodule is HC.
(Hence, classifying all HC-bimodules should be intractable.)

Let $\mathcal{B} \in HC(\mathcal{A})$, pick good filtration $\rightsquigarrow$ $\mathrm{gr}\,\mathcal{B}$
$\rightsquigarrow$ $\mathrm{Supp}(\mathrm{gr}\,\mathcal{B})$ = subvariety in X defined by $\mathrm{Ann}_A(\mathrm{gr}\,\mathcal{B})$

Exercise: • By an earlier remark, show that $\mathrm{Supp}(\mathrm{gr}\,\mathcal{B})$ is independent of the good filtration.
(Hence, we can simply call it $\mathrm{Supp}(\mathcal{B})$.
• (Harder) I = left annihilator of $\mathcal{B} \subseteq \mathcal{A}$. Then, $\mathrm{Supp}(\mathcal{A}/I) = \mathrm{Supp}(\mathcal{B})$.
Hint: $\mathcal{A}/I \curvearrowright \mathcal{B} \curvearrowleft \mathcal{A} \rightsquigarrow \mathcal{A}/I \hookrightarrow \underbrace{\mathrm{End}_{\mathcal{A}^{op}}(\mathcal{B})}$

has a good filtration induced by that on $\mathcal{B}$ and $\mathrm{Supp}(\mathrm{End}_{\mathcal{A}^{op}}(\mathcal{B})) \subseteq \mathrm{Supp}(\mathcal{B})$

• If $\mathcal{A}$ is simple, $\mathrm{Supp}(\mathcal{B}) = X$ for $\mathcal{B} \neq 0$.

Def^n: The category of HC bimodules with full support is the Serre quotient:
$$\overline{HC}(\mathcal{A}) = HC(\mathcal{A}) / \{\mathcal{B} : \mathrm{Supp}\,\mathcal{B} \subsetneq X\}.$$
It turns out that $\overline{HC}(\mathcal{A}) \simeq \mathrm{Rep}.$ (finite group)
as symmetric monoidal categories

$\pi_1^{alg.}(X^{reg})$ ← algebraic fundamental group

Controls finite étale covers of X^{reg}

$\pi_1^{alg}(X^{reg}) = \varprojlim \pi_1(X^{reg.}) /$ finite index normal subgroups
Fact: (Namikawa) $\pi_1^{alg}(X^{reg})$ is finite, and thus, equal to the maximal finite quotient of $\pi_1(X^{reg.})$.

Example: 1) $G \longrightarrow$ simply connected, semisimple

$\tilde{\mathbb{O}} = G/H$. Then, $\pi_1^{alg.}(\tilde{\mathbb{O}}) = H/H^\circ = \pi_1(\tilde{\mathbb{O}})$

2) $X = V/\Gamma$, $\Gamma \subseteq Sp(V)$ is finite.

$V \hookrightarrow$ symplectic vector space

Then, $\pi_1^{alg.}(X^{reg.}) = \pi_1(X^{reg}) = \Gamma$.

This is because $X^{reg.} = \underbrace{\{v \in V : \Gamma_v = 1\}}_{\shortparallel}/\Gamma$

$V \setminus \bigcup_{\text{finite}}$ symplectic subspaces $\leftarrow$ simply connected

For general X, let $\Gamma := \pi_1^{alg.}(X^{reg})$

<u>Classification Theorem</u>: (Losev '18) $\mathcal{HC}(\mathcal{A}) \xrightarrow{\sim} Rep(\Gamma/\Gamma_{\mathcal{A}})$.

$\Gamma_{\mathcal{A}} \triangleleft \Gamma$, under minor restrictions on X, one can compute $\Gamma_{\mathcal{A}}$ from the quantization parameter $\lambda \in \mathfrak{h}_X$.

<u>Example</u>: $\mathcal{A} = \mathcal{A}_0$ (canonical quantization)

The cover $\tilde{X}$ of $X^{reg.}$ corresponding to $\Gamma_{\mathcal{A}_0} \triangleleft \Gamma$ is as follows:

$\tilde{X} = Spec(\mathbb{C}[\tilde{X}])$ is a ramified cover of X.

Recall $\mathcal{L}_i \rightarrow$ all codimension 2 symplectic leaves of X.

$\rightsquigarrow X' = X^{reg} \sqcup \bigsqcup_{i=1}^{k} \mathcal{L}_i$

- open with $\operatorname{codim}_X X \setminus X' \geq 4$

$\tilde{X}$ is maximal s.t. $\tilde{X} \twoheadrightarrow X$ is unramified over X'.

Let $\Gamma' = \Gamma/\Gamma_{\mathcal{A}_0} \rightsquigarrow \Gamma' \curvearrowright \tilde{X} \xrightarrow[/\Gamma']{} X$

$\tilde{X}$ is conical singular symplectic $\rightsquigarrow$ canonical quantization

$\tilde{\mathcal{A}}_0 \curvearrowleft \Gamma'$ and $\tilde{\mathcal{A}}_0^{\Gamma'} \simeq \mathcal{A}_0$.

$\tau \in \mathrm{Rep}\,\Gamma' \rightsquigarrow \mathcal{B}_\tau = (\tau \otimes \tilde{\mathcal{A}}_0)^{\Gamma'}$

- HC - bimodule $\tilde{\mathcal{A}}_0^{\Gamma'} = \mathcal{A}_0$ - bimodule

$\tau \longmapsto \mathcal{B}_\tau$ is the map $\mathrm{Rep}(\Gamma') \xrightarrow{\sim} \overline{HC}(\mathcal{A}_0)$.

Application: Unipotent HC $\mathcal{U}\mathfrak{g}$ - bimodules

$\updownarrow$ ~ orbit method

Nilpotent orbits and covers

Theorem / Def^n: (LMBM '21, MBM '21) For a nilpotent cover $\tilde{\mathbb{O}}$ and its canonical quantization, $\mathcal{A}_0$ is a s.s. $\mathcal{U}\mathfrak{g}$-bimodule. The simple constituents are unipotent HC bimodules associated to $\tilde{\mathbb{O}}$.

Can: • Describe kernels of $\mathcal{U}\mathfrak{g} \longrightarrow \tilde{\mathcal{A}}_0$ (i.e. compute corresponding elements of $\mathfrak{h}^*/W$)

• Classify unipotent bimodules corresponding to $\tilde{\mathbb{O}}$ $\xleftrightarrow{\sim}$ Irreps. of finite group

Most of these bimodules are unitarizable.

Techniques involved in the proof of the classification theorem:

Step 1: Produce full monoidal embedding:

$$\mathcal{HC}(\mathcal{A}) \longrightarrow Rep(\Gamma).$$

$\rightsquigarrow \exists!\, \Gamma_{\mathcal{A}} \trianglelefteq \Gamma$ s.t. image $= Rep(\Gamma/\Gamma_{\mathcal{A}})$.

What does this embedding do on objects?

Let $\mathcal{B} \in \mathcal{HC}(\mathcal{A})$ and pick a good filtration.

$\rightsquigarrow gr\,\mathcal{B} \in A\text{-mod}.$

$gr\,\mathcal{B}$ comes with $\{\cdot,\cdot\} : A \times B \longrightarrow B$

$$\{a + \mathcal{A}_{\leq i-1},\ b + \mathcal{B}_{\leq j-1}\} \subseteq [a,b] + \mathcal{B}_{\leq i+j-d-1}$$

So, $gr\,\mathcal{B}$ becomes a f.g. Poisson A-module.

$\rightsquigarrow (gr\,\mathcal{B})|_{X^{reg}}$ - Poisson coherent sheaf on $\underset{\text{smooth, symplectic}}{\underset{\downarrow}{X^{reg}}}$.

<u>Fact</u>: $\exists!\, \mathcal{D}_{X^{reg}}$-module structure on a Poisson $\mathcal{O}_{X^{reg}}$-module lifting the $\mathcal{O}$-module structure, where $\{f,\cdot\} \in \mathcal{D}_{X^{reg}}$ ($f \in \mathcal{O}_{X^{reg}}$) acts by $\{f,\cdot\}$ coming from the Poisson module structure.

$gr(\mathcal{B})|_{X^{reg}}$ is an $\underbrace{\mathcal{O}\text{-coherent } \mathcal{D}\text{-module}}_{\parallel}$

vector bundle with flat connection

Pick $x \in X^{reg}$ $\rightsquigarrow$ monodromy representation

$$\pi_1(X^{reg}, x) \curvearrowright (gr\,\mathcal{B})_x$$

factors through $\Gamma = \pi_1^{alg}(X^{reg})$ ($\twoheadleftarrow \pi_1(X^{reg})$) because Γ is the maximal finite quotient.

So, once we pick a good filtration for $\mathcal{B}$, we get a
a Γ-rep. on $(\mathrm{gr}\,\mathcal{B})_x$. This is exactly the description
of the functor above on the level of objects.

Step 2: How to determine $\Gamma_{\mathcal{A}}$?

$\mathcal{A} \rightsquigarrow$ restrict to $X^{\mathrm{reg.}}$ to get a 'microlocal' sheaf
of filtered algebras on $X^{\mathrm{reg.}}$, denoted by $\mathcal{A}^{\mathrm{reg.}}$

$V \in \mathrm{Rep}\,\Gamma \rightsquigarrow$ vector bundle with flat connection on
$X^{\mathrm{reg.}}$ that uniquely quantizes to a sheaf of $\mathcal{A}^{\mathrm{reg}}$-
bimodules $\mathcal{B}_V^{\mathrm{reg}}$.

V lies in the image of $HC(\mathcal{A}) \iff \mathcal{B}_V^{\mathrm{reg.}}$ = microlocalisation
of an HC $\mathcal{A}$-bimodule ($\iff \mathcal{B}_V^{\mathrm{reg.}}$ extends to X.)

It may happen that $\mathcal{B}_V^{\mathrm{reg.}}$ doesn't extend (even that
$\Gamma(\mathcal{B}_V^{\mathrm{reg.}}) = 0$). Equivalently, the pushforward of $\mathcal{B}_V^{\mathrm{reg.}}$
to X may not be coherent.

$$X' = X^{\mathrm{reg.}} \amalg \coprod_{i=1}^{k} \mathcal{L}_i \quad , \quad \iota : X^{\mathrm{reg}} \hookrightarrow X'.$$

If $\iota_* \mathcal{B}_V^{\mathrm{reg}}$ is coherent, $\Gamma(\iota_* \mathcal{B}_V^{\mathrm{reg}})$ is the required
extension. Checking that $\mathcal{B}_V^{\mathrm{reg}}$ extends nicely to $\mathcal{L}_i$
reduces to a question about the transverse slice Σ_i
to $\mathcal{L}_i$, where Σ_i is a neighbourhood of 0 in $\mathbb{C}^2/\Gamma_i$,
$\lambda \longrightarrow$ parameters of $\mathcal{A}$ in $\mathfrak{h}_X = \bigoplus_{j=0}^{k} \mathfrak{h}_j$ with

$\mathfrak{h}_i \hookrightarrow \mathfrak{h}_{\Gamma_i} = \mathfrak{h}_{\mathbb{C}^2/\Gamma_i}$, $\lambda_i = \mathfrak{h}_i$ - component of λ.

$\rightsquigarrow$ The quantization $\mathcal{A}_i$ of $\mathbb{C}[\mathbb{C}^2/\Gamma_i]$ with parameter λ_i.

$\Gamma_i \longrightarrow \Gamma$ via $\Sigma_i \hookrightarrow X \rightsquigarrow \Sigma_i \setminus \{0\} \hookrightarrow X^{reg.}$

$\rightsquigarrow \phi_i : \Gamma_i = \pi_1^{alg.}(\Sigma_i \setminus \{0\}) \longrightarrow \pi_1^{alg.}(X^{reg.}) = \Gamma$.

Observation: $\mathcal{B}_V^{reg.}$ extends nicely to $\mathcal{L}_i \iff \phi_i^*(V)$ $\cap$ $\mathrm{Im}(HC(\mathcal{A}_i))$

This can be described as long as Γ_i is not E_8.

(E_8 is the only non-solvable Kleinian group.)

Generalization: Quantizing singular Lagrangians.

$X \longrightarrow$ conical symplectic singularity. A conical singular Lagrangian in X is $Y \subseteq X$ s.t.

0) Y is closed and $\mathbb{C}^\times$-stable

1) $Y \cap X^{reg}$ is Lagrangian (half dimensional, $\omega^{reg}|_{Y \cap X^{reg}} = 0$)

2) $\forall$ leaves $\mathcal{L} \subset X \implies Y \cap \mathcal{L}$ is isotropic in $\mathcal{L}$.

3) $\overline{Y \cap X^{reg}} = Y$

Favorable property: (That we'll assume on Y)

($\heartsuit$) $\mathrm{codim}_Y Y^{sing} \geq 2$ (Y is irreducible)

Example : $X \times X^{opp}$ $\quad X^{opp} \to X$ with $\{\cdot,\cdot\}$ mult. by -1

$\cup$

X_{diag}.

Then, X_{diag} is singular Lagrangian satisfying (♡).

Question : Quantize Y with additional structure.

If (♡) holds, the addition structure is a twisted local system (= vector bundle with twisted flat connection) on Y^{reg}.

The result of quantization is an $\mathcal{A}$-module $\mathcal{M}$ s.t. $\mathrm{supp}(\mathcal{M}) = Y$ and $\mathrm{gr}(\mathcal{M})|_{Y^{reg}}$ = chosen twisted local system

This classification question should reduce (under ♡) to the case when $\dim(X) = 4$ and $\dim(Y) = 2$.

Using this, Losev and Shilin Yu have classified irred. HC $(\mathfrak{g}, K)$-modules with full support over quantizations of $\mathbb{C}[\mathbb{O}]$ s.t. $\mathrm{codim}_{\mathbb{O}} \overline{\mathbb{O}} \setminus \mathbb{O} \geq 4$.