(joint with Daniele Rosso)

Schur - Weyl duality

Let $V = \mathbb{C}^n$ $\qquad GL(V) = GL_n \qquad SL_n = SL(V)$

$S_d \longrightarrow$ symmetric group on d letters

$$GL(V) \curvearrowright V^{\otimes d} \curvearrowleft S_d$$

These two actions centralize each other.

$$\rightsquigarrow \quad V^{\otimes d} \simeq \bigoplus_{\substack{\lambda \vdash d \\ \ell(\lambda) \leq n}} L_\lambda \boxtimes M_\lambda$$

$\lambda \longrightarrow$ partitions of d with at most n parts

Quantum version

Let $V = (\mathbb{C}(v))^n$

$$U_v(sl_n) \curvearrowright V^{\otimes d} \curvearrowleft H_d$$

$U_v(sl_n)$: Quantized enveloping algebra generated by $e_i, f_i, k_i^{\pm 1}$ for $1 \leq i \leq n-1$:

$$k_i k_j = k_j k_i$$

$$k_i e_j k_i^{-1} = v^{a_{ij}} e_j$$

$$k_i f_j k_i^{-1} = v^{-a_{ij}} e_j$$

$$[e_i, f_j] = \delta_{i,j} \frac{k_i - k_i^{-1}}{v - v^{-1}}$$

$$\left.\begin{aligned} e_i e_j &= e_j e_i \\ f_i f_j &= f_j f_i \end{aligned}\right\} \text{ if } |i-j| \neq 1$$

$$\left.\begin{aligned} (v+v^{-1})\, e_i e_j e_i &= e_i^2 e_j + e_j e_i^2 \\ (v+v^{-1})\, f_i f_j f_i &= f_i^2 f_j + f_j f_i^2 \end{aligned}\right\} \text{ if } |i-j| = 1$$

H_d: Hecke algebra of Type A

$$\tau_1, \dots, \tau_{d-1}$$

$$\tau_i^2 = (v^2-1)\tau_i + v^2$$

$$\tau_i \tau_j = \tau_j \tau_i \qquad \text{if } |i-j| \neq 1$$

$$\tau_i \tau_j \tau_i = \tau_j \tau_i \tau_j \qquad \text{if } |i-j| = 1.$$

$$V^{\otimes d} \simeq \bigoplus_{\substack{\lambda \vdash d \\ \ell(\lambda) \leq n}} L_\lambda \boxtimes M_\lambda$$

<u>Convolution</u>

X = set of complete flags in $\mathbb{F}_q^d$

$= \{ F = (0 \subset F_1 \subset F_2 \subset \dots \subset F_{d-1} \subset F_d = \mathbb{F}_q^d) :$

$\dim F_i = i \}.$

$GL_d(\mathbb{F}_q) \curvearrowright X$.

Define a product on $\mathbb{C}(X\times X)$: Given $\alpha, \beta \in \mathbb{C}(X\times X)$, define

$$\alpha * \beta(F, F') = \sum_{F''\in X} \alpha(F, F'')\,\beta(F'', F')$$

This descends to a product on $\mathbb{C}(X\times X)^G$.

$$\mathrm{Dim}(\mathbb{C}(X\times X)^G) = n!$$

$$(X\times X)/G \longleftrightarrow S_n$$

$$(F, F') \longmapsto a_{ij} = \dim\left(\frac{F_i\cap F_j'}{F_i\cap F_{j-1}' + F_{i-1}\cap F_j'}\right)$$

Iwahori: $\mathbb{C}(X\times X)^G \simeq H_d\big|_{v=q^{1/2}}$

Similarly, if Y = set of n-step partia flags in $\mathbb{F}_q^d$

$= \{(0 = F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n = \mathbb{F}_q^d)$

Beilinson - Lusztig - McPherson: the convolution algebra $\mathbb{C}(Y\times Y)^G \simeq$ quantum Schur algebra $\big|_{v=q^{1/2}}$

$=$ image of $U_v(sl_n)$ in $\mathrm{End}(\mathbb{C}(v)^{\otimes d})$.

Finally, $\mathbb{C}(Y\times X)^{G} \simeq \mathbb{C}(V)^{\otimes d}\big|_{v=q^{1/2}}$

Then, quantum Schur-Weyl duality:

$$\mathbb{C}(Y\times Y)^{G} \curvearrowright \mathbb{C}(Y\times X)^{G} \curvearrowleft \mathbb{C}(X\times X)^{G}$$

(Lusztig - Grojnowski)

Mirabolic subgroup

$M \subseteq GL(V)$

$\|$

Stabilizer of some $v \in V\setminus\{0\}$.

$\cap$

$$\begin{pmatrix} \boxed{} \\ 0 \cdots 0 \ 1 \end{pmatrix}$$

For any set X with a $GL(V)$-action

M-orbits on X $\longleftrightarrow$ $GL(V)$-orbits on $X\times(V\setminus 0)$

Mirabolic quantum Schur-Weyl duality

For any finite set X, we can define a convolution product on the space

$$\mathbb{C}(X\times X\times \mathbb{F}_q^{\,d})^{G}$$

$$\alpha * \beta(F,F',v) := \sum_{(F'',u)} \alpha(F,F'',u)\,\beta(F'',F',v-u)$$

$\mathbb{C}(X \times X \times \mathbb{F}_q^d) \longrightarrow$ Mirabolic Hecke algebra MH_d

(Solomon '91, Siegel '93)

$\mathbb{C}(Y \times Y \times \mathbb{F}_q^d) \longrightarrow$ Mirabolic quantum Schur algebra (Rosso '18)

$\mathbb{C}(Y \times X \times \mathbb{F}_q^d) \longrightarrow$ Mirabolic tensor space. $MV_{n,d}$

$$MU(n) \curvearrowright MV_{n,d} \curvearrowleft MH_d .$$

(Rosso '18, Fan-Zhang-Ma '25)

$\rightsquigarrow$

Presentation of $MU(n)$ in terms of generators and relations, formulae for actions on $MV_{n,d}$.

Mirabolic quantum group

$\mathbb{C}(v)$ - algebra generated by $e_i, f_i, k_i^{\pm 1}, \ell$ satisfying the same relations as before $1 \leq i \leq n-1$ and :

$$\ell^2 = \ell$$

$$\ell k_i = k_i \ell$$

$$\ell e_i = e_i \ell \, , \quad \ell f_i = f_i \ell \qquad \text{if } i > 1$$

$$\ell e_1 \ell = \ell e_1 \, , \quad \ell f_1 \ell = f_1 \ell$$

$$(v+v^{-1})\, e_1 \ell e_1 = v^{-1} e_1^2 \ell + v \ell e_1^2$$
$$(v+v^{-1})\, f_1 \ell f_1 = v f_1^2 \ell + v^{-1} \ell f_1^2 .$$

Recall : $U_v(sl_n)$ is actually a Hopf algebra.

$$\Delta(e_i) = 1 \otimes e_i + e_i \otimes k_i$$
$$\Delta(f_i) = k_i^{-1} \otimes f_i + f_i \otimes 1$$
$$\Delta(k_i^{\pm 1}) = k_i^{\pm 1} \otimes k_i^{\pm 1}$$

Proposition : $MU(n)$ is a co-module algebra over $U_v(sl_n)$.

$$\rho : MU(n) \longrightarrow U_v(sl_n) \otimes MU(n)$$
$$\rho(\ell) = 1 \otimes \ell$$

Consequently, if $L \in U_v(sl_n)$-mod and $W \in MU(n)$-mod, then $L \otimes W \in MU(n)$-mod.

$V = \mathbb{C}(v)^n \curvearrowleft U_v(sl_n)$

Define for $0 \le r \le n$,

$U_v(sl_n) \curvearrowright W_r = \Lambda^r V \subseteq V^{\otimes r}$

spanned by $v_{i_1} \wedge v_{i_2} \wedge \dots v_{i_r}$

s.t. $i_1 < i_2 < \dots < i_r$

We get an $MU(n)$-action by :

$$l \cdot (v_{i_1} \wedge v_{i_2} \cdots \wedge v_{i_n}) = \begin{cases} 0 & \text{if } i_1 = 1 \\ v_{i_1} \wedge v_{i_2} \cdots \wedge v_{i_n} & \text{if } i \neq 1 \end{cases}$$

Theorem:

- $L_\lambda \otimes W_r$ is a f.d. simple $MU(n)$-representation for any dominant integral weight λ and $0 \le r \le n$.
- This is a complete list of all simple representations.
- $MU(n)$-mod is semisimple.

Main ideas of proof:

- Mirabolic weight spaces

$$M = \bigoplus M_{\mu, \varepsilon} \quad \text{where}$$

$$M_{\mu,\varepsilon} = \left\{ m \in M : \begin{array}{l} k_i \cdot m = v^{\mu_i - \mu_{i+1}} m \\ l \cdot m = \varepsilon m \end{array} \right\}$$

The weight space decomposition of a module uniquely determines it upto isomorphism.

- Mirabolic Verma modules

Define $V_{\lambda, r} := V_\lambda \otimes W_r$

We determine I such that $MU(n)/_I \simeq V_\lambda \otimes W_r$.

MH_d-mod - semisimple and irreducibles $\longleftrightarrow$ $(\lambda, 1^r)$ such that $|\lambda| + r = d$.

<u>Theorem</u>: $$MV_{n,d} = \bigoplus_{\substack{\lambda, r \\ |\lambda| + r = d \\ \ell(\lambda) \leq n,\, r \leq n}} L_{\lambda, r} \otimes M^{(\lambda, 1^r)}$$

The proof involves a careful analysis of the mirabolic weight spaces and mirabolic Jucys-Murphy elements in MH_d.