Pigeonhole Principle

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1. In a tournament between n teams where each team plays every other team exactly once, resulting in either a win or a loss, show that if each team wins at least once, there exist two teams that have exactly the same number of wins.

2. Prove that in a group of 6 people, there always exist 3 mutual acquaintances or 3 mutual strangers.

3. If 10 points are chosen at random in the interior of an equilateral triangle with side length 3, prove that there exists a pair of points that are at distance at most 1 from each other.

4. Given 5 lattice points in the plane, show that there exists a pair of points whose mid-point is also a lattice point.

5. Consider a subset A of the set $\{1, 2, \ldots 2n\}$ of size $n+1$.
   - Prove that there exists a relatively prime pair in A.
   - Prove there exists a factor-multiple pair in A.

6. Given a sequence of integers $a_1, a_2, \ldots, a_n$, prove that there exist a subsequence of consecutive integers $a_{k+1}, a_{k+2}, \ldots, a_l$ such that $\sum\limits_{i=k+1}^{l} a_i$ is divisible by n.

7. Show that, given a 7-digit number, you can cross out some digits at the beginning and at the end such that the remaining number is divisible by 7.

8. Consider any natural number n that is co-prime to 10. Show that there exists a number in the sequence $1, 11, 111, 1111, 11111, \ldots$ which is divisible by n.

9. Let $n \geq 3$ be an odd number. Show that at least one number in the set $\{2^1 - 1, 2^2 - 1, \cdots, 2^{n-1} - 1\}$ is divisible by n.

10. (Dutch Mathematics Olympiad) A set S of positive integers is called square-free if for all distinct $a, b \in S$ we have that the product ab is not a square. What is the maximum cardinality of a square free subset $S \subseteq \{1, 2, 3, ..., 25\}$?

11. Prove that ever set of of 10 distinct integers between 1 and 100 contains two non-empty disjoint subsets such that their sum is equal.

12. Given any 3 distinct integers, there exists a pair x and y such that $F(x, y) \equiv x^3y - xy^3$ is divisible by 30.

13. How many (a) bishops (b) rooks (c) knights (d) kings (e) queens can one put on an 8×8 chessboard such that no two can hit each other?

14. Given an 8×8 chessboard with 33 rooks placed, show that 5 can be chosen that do no hit each other.

15. A team played 20 games over a period of 15 says such that at least one game was played every day. Show that there was a period of consecutive days during which exactly 9 games were played.

16. (Dirichlet's Approximation Theorem) Show that for any irrational $x \in \mathbb{R}$ and positive integer n, there exists a rational number $\frac{p}{q}$ with $1 \leq q \leq n$ such that $|x - \frac{p}{q}| < \frac{1}{nq}$.

17. Every point in the plane is colored either red, green, or blue. Prove that there exists a rectangle in the plane such that all four of its vertices are the same color.

18. Show that in any given set A of 13 distinct real numbers, there are at least two numbers x and y such that
$$0 < \frac{x-y}{1+xy} \leq 2 - \sqrt{3}.$$

19. Given a sequence of $mn+1$ distinct real numbers, prove that there either exists an increasing subsequence of length $n+1$ or a decreasing sequence of length $m+1$. Check that the result isn't true if $mn+1$ is replaced by mn.

20. We have mn people standing in an $m \times n$ array such that their heights are non-decreasing in each row from left to right. Now, suppose the people in every column shuffle their order such that their heights are non-decreasing from front to back. Show that in the new arrangement of people, heights are still non-decreasing in each row from left to right.

21. (INMO 2011) Suppose five of the nine vertices of a regular nine-sided polygon are arbitrarily chosen. Show that one can select four among these five such that they are the vertices of a trapezium.

22. (Dutch Mathematical Olympiad) Suppose that S is a subset of $\{1, 2, 3, ..., 30\}$ with at least 11 elements. Show that one can choose a nonempty subset T of S such that the product of all elements of T is a square.

23. A binary word of length n is a sequence of 0's and 1's of length n. The set $\{0,1\}^n$ is the set of all binary words of length n. Let S be a subset of $\{0,1\}^n$ with the following property: for every pair of distinct elements $x = x_1x_2 \cdots x_n$ and $y = y_1y_2 \cdots y_n$ we have that x and y differ in at least 3 positions. Show that S has at most $\frac{2^n}{n+1}$ elements.

24. (IMO 1987) Let $x_1, x_2, ..., x_n$ be real numbers satisfying $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Prove that for every integer $k \geq 2$ there are integers $a_1, a_2, \ldots, a_n$, not all 0, such that $|a_i| \leq k-1$ for all i and
$$|a_1x_1 + a_2x_2 + \cdots + a_nx_n| \leq \frac{(k-1)\sqrt{n}}{k^n - 1}.$$