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ABSTRACT

Losev introduced the scheme X of almost commuting elements (i.e., elements commuting upto a rank one element) of $\mathfrak{g} = \mathfrak{sp}(V)$ for a symplectic vector space V and discussed its algebro-geometric properties. We construct a Lagrangian subscheme X^{nil} of X and show that it is a complete intersection of dimension $\dim(\mathfrak{g}) + \frac{1}{2}\dim(V)$ and compute its irreducible components.

We study the quantum Hamiltonian reduction of the algebra $\mathcal{D}(\mathfrak{g})$ of differential operators on the Lie algebra $\mathfrak{g}$ tensored with the Weyl algebra with respect to the action of the symplectic group, and show that it is isomorphic to the spherical subalgebra of a certain rational Cherednik algebra of Type C. We contruct a category $\mathcal{C}_c$ of $\mathcal{D}$-modules whose characteristic variety is contained in X^{nil} and construct an exact functor from this category to the category $\mathcal{O}$ of the above rational Cherednik algebra. Simple objects of the category $\mathcal{C}_c$ are mirabolic analogs of Lusztig's character sheaves.

We also define and study a group-theoretic version of Losev's almost commuting scheme as well as the above quantum Hamiltonian reduction problem.

CHAPTER 1

INTRODUCTION

1.1 Main results of the thesis

Let $V := \mathbb{C}^{2n}$ be a symplectic vector space and let $\mathfrak{g}$ denote the Lie algebra $\mathfrak{sp}(V) = \mathfrak{sp}_{2n}$. The almost commuting scheme X of $\mathfrak{g}$ was defined by Losev in [Los21] as the closed subscheme of $\mathfrak{g} \times \mathfrak{g} \times V$ defined by the ideal I generated by the matrix entries of $[x, y] + i^2$, i.e., by all functions of the form $(x, y, i) \mapsto \lambda([x, y] + i^2)$ for $\lambda \in \mathfrak{g}^*$. Here, we use the fact that $\mathrm{Sym}^2(V)$ can be identified with $\mathfrak{sp}(V)$ to view i^2 as an element of $\mathfrak{sp}(V)$ (see, for example, [CG10, Lemma 1.3.5]). The geometrical properties of X were studied by Losev who showed that:

Theorem 1.1.1 ([Los21])**.** *The scheme X is reduced, irreducible and a complete intersection of dimension $2n^2 + 3n = \dim(\mathfrak{g}) + \dim(V)$.*

In this work, we consider the reduced subscheme X^{nil} of X defined as:

$$X^{nil} := \{(x, y, i) \in \mathfrak{g} \times \mathfrak{g} \times V : [x, y] + i^2 = 0 \text{ and } y \text{ is nilpotent}\}.$$

This definition is motivated by the notion of character sheaves, first defined by Lusztig (see [Lus85, Lus91]). It was shown by Mirković and Vilonen [MV88] and Ginzburg [Gin89] that over $\mathbb{C}$, a character sheaf on a reductive algebraic group K can be defined as an $\mathrm{Ad}(K)$-equivariant perverse sheaf M on K, such that the corresponding characteristic variety lies in the nilpotent locus $K \times \mathcal{N} \subseteq K \times \mathfrak{k}^*$, where $\mathfrak{k} = \mathrm{Lie}(K)$ and $\mathcal{N} \subseteq \mathfrak{k}^* \simeq \mathfrak{k}$ is the nilpotent cone. Constructions analogous to X^{nil} were done in [GG06, FG10a, FG10b] to provide 'mirabolic' analogs of these character sheaves in Type A.

We describe some notation. It is known (for example, see [CM93, Theorem 5.1.3]) that nilpotent conjugacy classes in $\mathfrak{g}$ are parametrized by the partitions λ of $2n$ in which every odd part appears an even number of times. Let P_n be the set of all such partitions and let $\mathcal{P}_n$ denote the subset of those partitions in P_n in which all the parts are even. For each $\lambda \in P_n$, let $\mathcal{N}_\lambda$ denote the corresponding nilpotent conjugacy class in $\mathfrak{g}$. Define for each $\lambda \in P_n$:

$$X_\lambda := \{(x, y, i) \in X^{nil} : y \in \mathcal{N}_\lambda\}.$$

Let $\overline{X_\lambda}$ denote the closure of X_λ in X^{nil}.

Note that we can identify $\mathfrak{g} \times \mathfrak{g} \times V$ with $T^*(\mathfrak{g}) \times V$ using the trace form on $\mathfrak{g}$. This gives $\mathfrak{g} \times \mathfrak{g} \times V$ a natural symplectic structure. Our first main result reads:

Theorem 1.1.2. *1. The scheme X^{nil} is a complete intersection in $\mathfrak{g} \times \mathfrak{g} \times V$ of dimension $2n^2 + 2n$. The irreducible components of X^{nil} are exactly given by the $\overline{X_\lambda}$ for $\lambda \in \mathcal{P}_n$.*

2. With the standard symplectic structure, X^{nil} is a Lagrangian subscheme of $\mathfrak{g} \times \mathfrak{g} \times V$.

A similar Lagrangian subscheme was constructed in [GG06] in the context of the almost commuting scheme of the Lie algebra $\mathfrak{gl}_n$. Using Theorem 1.1.2, we provide an independent proof of Theorem 1.1.1 in the style of [GG06], that eliminates the use of results from [Los06]. Furthermore, the scheme X^{nil} will be used in §4.2.4 to provide a mirabolic analog of Lusztig's character sheaves in Type C.

We next discuss some Hamiltonian reduction problems arising in the context of the scheme X and some other related schemes. For this, we define the following subschemes of $\mathfrak{g} \times \mathfrak{g} = \mathrm{Spec}(\mathbb{C}[\mathfrak{g} \times \mathfrak{g}]) = \mathrm{Spec}(\mathbb{C}[x, y])$. Consider the commuting scheme C which is the (not necessarily reduced) subscheme of $\mathfrak{g} \times \mathfrak{g}$ defined by the ideal of $\mathbb{C}[x, y]$ generated by the matrix entries of the commutator $[x, y]$. Next, define the scheme A to be the (not necessarily reduced) subscheme of $\mathfrak{g} \times \mathfrak{g}$ defined by the ideal of $\mathbb{C}[x, y]$ generated by all the 2×2 minors of the commutator $[x, y]$.

Note that the set of $\mathbb{C}$-points of the underlying reduced subscheme of C consists of pairs of elements of $\mathfrak{g}$ that commute with each other, whereas that of A consists of pairs of elements of $\mathfrak{g}$ whose commutator has rank lesser than or equal to one. The commuting scheme C is of wide interest, and its geometrical properties (most notably, its reducedness) are largely unknown. It is known that C is irreducible (see [Ric79]).

The schemes X, C and A have an action of the symplectic group $G = Sp(V)$ obtained by the adjoint action on $\mathfrak{g}$ and the natural action on V. Hence, we can consider the respective categorical quotients of these schemes by the action of G. While it isn't known if C is reduced, it was shown in [Los21] that there's an isomorphism:

$$C//G \longrightarrow X//G,$$

which, paired with Theorem 1.1.1, implies that $C//G$ is reduced. (That $C//G$ is reduced was deduced independently in [CN21] slightly earlier, by proving a version of the Chevalley restriction theorem for the commuting scheme of $\mathfrak{g}$.) We extend this isomorphism to show that $C//G \simeq X//G \simeq A//G$. In fact, we prove the following stronger result:

Theorem 1.1.3. *We have an isomorphism of schemes:*

$$X//\{\pm 1\} \longrightarrow A//\{\pm 1\} = A,$$

where $\{\pm 1\} \subseteq G$ is the center of the symplectic group. In particular, the scheme A is reduced.

An analog of the isomorphism $X//G \simeq A//G$ for the Lie algebra $\mathfrak{gl}_n$ was proved in [GG06]. The theorem is deduced from a linear algebraic lemma (Lemma 4.1.4), which also implies an algebro-geometric analog of the 'shifting trick' in the theory of Hamiltonian reduction, that is well-known in the differential-geometric setting (see [GS82], [CS91]).

The above categorical quotients can (and will) all be viewed as classical Hamiltonian reductions of certain schemes under the action of the group G:

- The scheme $X//G$ is the reduction of the scheme $\mathfrak{g} \times \mathfrak{g} \times V$ with respect to G at 0.

- The scheme $C//G$ is the reduction of the scheme $\mathfrak{g} \times \mathfrak{g}$ with respect to G at 0.

- The scheme $A//G$ is the reduction of the scheme $\mathfrak{g} \times \mathfrak{g}$ with respect to G at the closure of the orbit of rank 1 matrices in $\mathfrak{g} \simeq \mathfrak{g}^*$.

So, we can try to study the non-commutative or quantum analogs of these reduction problems. For this, let $\mathcal{U}\mathfrak{g}$ denote the universal enveloping algebra of $\mathfrak{g}$, let $\mathcal{D}(\mathfrak{g})$ denote the algebra of polynomial differential operators on $\mathfrak{g}$ and let W_{2n} denote the Weyl algebra on $2n$ variables, which is the algebra of polynomial differential operators on the affine n-space. Then, $\mathcal{D}(\mathfrak{g})$ is a quantization of $\mathbb{C}[\mathfrak{g} \times \mathfrak{g}] \simeq \mathbb{C}[\mathfrak{g} \times \mathfrak{g}^*]$, whereas W_{2n} is a quantization of $\mathbb{C}[V]$. Both the algebras $\mathcal{D}(\mathfrak{g})$ and W_{2n} have a natural $\mathfrak{g}$-action (and, thus, so does their tensor product.) So, we get quantum co-moment maps:

$$\Theta_0 : \mathcal{U}\mathfrak{g} \longrightarrow \mathcal{D}(\mathfrak{g}),$$

$$\Theta_2 : \mathcal{U}\mathfrak{g} \longrightarrow \mathcal{D}(\mathfrak{g}) \otimes W_{2n}.$$

(These maps are elaborated upon in §4.2.1.) Then, we can consider the following non-commutative algebras:

- The reduction $\Big((\mathcal{D}(\mathfrak{g}) \otimes W_{2n})/(\mathcal{D}(\mathfrak{g}) \otimes W_{2n}) \cdot \Theta_2(\mathfrak{g})\Big)^{\mathfrak{g}}$ of $\mathcal{D}(\mathfrak{g}) \otimes W_{2n}$ at the augmentation ideal of $\mathcal{U}\mathfrak{g}$.

- The reduction $\Big(\mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g}) \cdot \Theta_0(\mathfrak{g})\Big)^{\mathfrak{g}}$ of $\mathcal{D}(\mathfrak{g})$ at the augmentation ideal of $\mathcal{U}\mathfrak{g}$.

- The reduction $\Big(\mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g}) \cdot \Theta_0(\mathcal{K})\Big)^{\mathfrak{g}}$ of $\mathcal{D}(\mathfrak{g})$ at the unique primitive ideal $\mathcal{K} \subseteq \mathcal{U}\mathfrak{g}$ such that $\mathrm{gr}(\mathcal{K}) \subseteq \mathbb{C}[\mathfrak{g}^*]$ is the defining ideal of the orbit of rank 1 matrices.

(Each of these algebras is discussed in detail in §4.2.1.) The algebra $\Big(\mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g}) \cdot \Theta_0(\mathfrak{g})\Big)^{\mathfrak{g}}$ has been studied classically by Harish-Chandra (see [HC64]) who constructed a surjective algebra homorphism called the ‘radial parts’ homomorphism:

$$\mathcal{D}(\mathfrak{g})^{\mathfrak{g}} \longrightarrow \mathcal{D}(\mathfrak{h})^{W},$$

where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and W is the Weyl group. The kernel of this homomorphism was shown to be precisely $(\mathcal{D}(\mathfrak{g}) \cdot \Theta_0(\mathfrak{g}))^{\mathfrak{g}}$ in the works of Wallach [Wal93] and Lavasseur and Stafford [LS95, LS96], implying that the algebra $\Big(\mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g}) \cdot \Theta_0(\mathfrak{g})\Big)^{\mathfrak{g}}$ is isomorphic to $\mathcal{D}(\mathfrak{h})^{W}$.

In this work, we'll discuss the other two quantum Hamiltonian reduction problems. For this, we recall the rational Cherednik algebra H_c of Type C, first defined in [EG02]. Here, the parameter $c = (c_{long}, c_{short})$ lies in $\mathbb{C}^2$. Let $e = \frac{1}{|W|}\sum_{w\in W} w$ be the averaging idempotent of the Weyl group W and consider the spherical subalgebra $eH_ce \subseteq H_c$ of the Cherednik algebra. (The notation is elaborated on in §2.5.)

We prove the following theorem about these algebras:

Theorem 1.1.4. *We have algebra isomorphisms:*

$$\Big((\mathcal{D}(\mathfrak{g}) \otimes W_{2n})/(\mathcal{D}(\mathfrak{g}) \otimes W_{2n}) \cdot \Theta_2(\mathfrak{g})\Big)^{\mathfrak{g}} \simeq \Big(\mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g}) \cdot \Theta_0(\mathcal{K})\Big)^{\mathfrak{g}} \simeq eH_ce,$$

for the parameter $c = (-1/4, -1/2)$.

Analogs of this theorem were proved in [EG02] and [GG06] in the $\mathfrak{gl}_n$-setting. This theorem shows that for the very special choice of the parameter $c = (-1/4, -1/2)$, the spherical subalgebra eH_ce can be obtained as a quantum Hamiltonian reduction of the ring of differential operators on $\mathfrak{g}$. The proof of the first isomorphism in the theorem follows from Lemma 4.1.4. The proof of the second isomorphism employs a generalization of the radial parts construction studied by Etingof and Ginzburg in [EG02].

Inspired by the formalism in [GG06, §7], we define a certain category $\mathcal{C}$ of holonomic $(\mathcal{D}(\mathfrak{g}) \otimes W_{2n})$-modules supported on X^{nil}. The simple objects of $\mathcal{C}$ will be the mirabolic analogs of Lusztig's character sheaves in Type C. We construct an exact functor from $\mathcal{C}$ to $\mathcal{O}(eH_ce)$, the category $\mathcal{O}$ of the spherical Cherednik algebra eH_ce defined in [BEG03a].

In the last part of this work, we study the group-theoretic analogs of the above problems. We define the group-theoretic almost commuting scheme $\mathfrak{X}$ to be the (not necessarily reduced) subscheme of $T^*(G) \times V \simeq G \times \mathfrak{g} \times V$ defined by the ideal generated by the matrix entries of $\mathrm{Ad}g(y) - y + i^2$ for $(g, y, i) \in G \times \mathfrak{g} \times V$. Also, let $\mathfrak{X}^{nil}$ be the reduced subcheme of $\mathfrak{X}$ obtained by stipulating that the element y is nilpotent. Then, we prove the following theorem about these schemes:

Theorem 1.1.5. *1. The scheme $\mathfrak{X}$ is an irreducible scheme that is a reduced, complete intersection in $G \times \mathfrak{g} \times V$ of dimension $2n^2 + 3n = \dim(G) + \dim(V)$.*

2. The scheme $\mathfrak{X}^{nil}$ is a Lagrangian complete intersection in $G \times \mathfrak{g} \times V$ of dimension $2n^2 + 2n = \dim(G) + \frac{1}{2}\dim(V)$.

Further, we can define group-theoretic versions of the schemes C and A and consider the classical Hamiltonian reduction problems arising in their context. We study these schemes in §5.1 where, in particular, we prove a group-theoretic analog of Theorem 1.1.3 (see Theorem 5.1.7).

Finally, we consider the quantum Hamiltonian reduction of the algebra of differential operators $\mathcal{D}(G)$ on the group G. This algebra has a natural $\mathfrak{g}$-action, and therefore, so does the tensor product $\mathcal{D}(G) \otimes W_{2n}$. Hence, we get quantum co-moment maps $\Sigma_0 : \mathcal{U}\mathfrak{g} \longrightarrow \mathcal{D}(G), \Sigma_2 : \mathcal{U}\mathfrak{g} \longrightarrow \mathcal{D}(G) \otimes W_{2n}$, and we can consider the non-commutative algebras:

- The reduction $\Big((\mathcal{D}(G) \otimes W_{2n})/(\mathcal{D}(G) \otimes W_{2n}) \cdot \Sigma_2(\mathfrak{g})\Big)^{\mathfrak{g}}$ of $\mathcal{D}(G) \otimes W_{2n}$ at the augmentation ideal of $\mathcal{U}\mathfrak{g}$.
- The reduction $\Big(\mathcal{D}(G)/\mathcal{D}(G) \cdot \Sigma_0(\mathfrak{g})\Big)^{\mathfrak{g}}$ of $\mathcal{D}(G)$ at the augmentation ideal of $\mathcal{U}\mathfrak{g}$.
- The reduction $\Big(\mathcal{D}(G)/\mathcal{D}(G) \cdot \Sigma_0(\mathcal{K})\Big)^{\mathfrak{g}}$ of $\mathcal{D}(G)$ at the unique primitive ideal $\mathcal{K} \subseteq \mathcal{U}\mathfrak{g}$ such that $\mathrm{gr}(\mathcal{K}) \subseteq \mathbb{C}[\mathfrak{g}^*]$ is the defining ideal of the orbit of rank 1 matrices.

(The above maps and notation are elaborated upon in §5.2.) The algebra $\Big(\mathcal{D}(G)/\mathcal{D}(G)\cdot \Sigma_0(\mathfrak{g})\Big)^{\mathfrak{g}}$ was considered by Harish-Chandra in [HC64] and it is known from [Wal93, LS95, LS96] that this algebra is isomorphic to $\mathcal{D}(H)^W$, where $H \subseteq G$ is a maximal torus.

We will study the other two algebras. We recall the degenerate version of Cherednik's double Affine Hecke algebra (see [Che05]), also known as the trigonometric Cherednik algebra, of Type C, denoted by H_c^{trig} for a parameter $c \in \mathbb{C}^2$. As in the rational case, we can define the spherical subalgebra $eH_c^{trig}e \subseteq H_c^{trig}$, where e is the averaging idempotent of the Weyl group W. We prove that:

Theorem 1.1.6. *We have algebra isomorphisms:*

$$\Big((\mathcal{D}(G)\otimes W_{2n})/(\mathcal{D}(G)\otimes W_{2n})\cdot \Sigma_2(\mathfrak{g})\Big)^{\mathfrak{g}} \simeq \Big(\mathcal{D}(G)/\mathcal{D}(G)\cdot \Sigma_0(\mathcal{K})\Big)^{\mathfrak{g}} \simeq eH_c^{trig}e,$$

for the parameter $c = (-1/4, -1/2)$.

1.2 Organization

Here, we give more details about the structure of this thesis.

In Chapter 2, we recall all the relevant definitions and notation from symplectic geometry that we will be using. We also note down results about rational and trigonometric Cherednik algebras that will be needed for our proofs.

In Chapter 3, we prove Theorem 1.1.2. The proof of the fact that X^{nil} is Lagrangian is by embedding it into the Lagrangian subscheme defined in [GG06]. The proof of the rest of the theorem will be seen to be a consequence of this fact and some elementary $\mathfrak{sl}_2$-theory.

In Chapter 4, we study the Hamiltonian reduction problems in the Lie algebraic setting. In §4.1, we prove Lemma 4.1.4 and use it to deduce Theorem 1.1.3. In §4.2.1, we define the algebras alluded to in the statement of Theorem 1.1.4. In §4.2.2 and §4.2.3, we construct maps between these algebras and prove that they are isomorphisms. Finally, in §4.2.4, we note some results about the category $\mathcal{C}$ and provide the construction of the functor from $\mathcal{C}$ to $\mathcal{O}(eH_ce)$.

In Chapter 5, we prove group theoretic versions of the theorems proved in Chapters 3 and 4. In particular, we prove Theorems 1.1.5 and 1.1.6 in §5.1 and §5.2 respectively.

We end with Appendix A where we prove that the ring of invariant polynomials $\mathbb{C}[H\times\mathfrak{h}]^W$ is generated as a Poisson algebra by its subalgebras $\mathbb{C}[H]^W$ and $\mathbb{C}[\mathfrak{h}]^W$. This is a group theoretic version of a result in [Wal93] and is needed for the proof of Theorem 1.1.6.

CHAPTER 2

PRELIMINARIES

In this chapter, we develop some preliminaries. We define all the important notions, state the known results and provide references for their proofs. We work over $\mathbb{C}$ throughout.

2.1 Symplectic geometry

Definition 2.1.1. A symplectic variety Y over $\mathbb{C}$ is one equipped with a non-degenerate algebraic 2-form ω such that $d\omega = 0$.

Example 2.1.1. The prototypical example of a symplectic variety is the cotangent bundle $T^*(X)$ of a smooth variety X. The symplectic structure on $T^*(X)$ is given by the 2-form $\omega = d\lambda$, where λ is the canonical 1-form on $T^*(X)$, also called the Liouville 1-form.

For $X = \mathbb{C}^n$, we can give an explicit description of the symplectic form ω on $T^*(X) \simeq \mathbb{C}^{2n}$. Given coordinates $q_1, q_2, \ldots, q_n$ on $\mathbb{C}^n$ and dual coordinates $p_1, p_2, \ldots, p_n$ on the cotangent space, the form ω is given via the formula:

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

Definition 2.1.2. A commutative algebra A over $\mathbb{C}$ is called a Poisson algebra if there exists a $\mathbb{C}$-linear bracket $\{\cdot, \cdot\} : A \times A \to A$ such that:

1. The bracket $\{\cdot, \cdot\}$ gives A a Lie algebra structure.

2. The Liebniz rule is satisfied: $\{fg, h\} = f\{g, h\} + g\{f, h\}$ for all $f, g, h \in A$.

Example 2.1.2. Given a symplectic variety Y, we have a natural Poisson algebra structure on the ring of regular functions $\mathcal{O}(Y)$, which we describe here. Let ω be the symplectic form on Y. As this form is assumed to be non-degenerate, we get a map from $\mathcal{O}(Y)$ to the space of vector fields on Y (i.e., derivations on $\mathcal{O}(Y)$) that maps an element $f \in \mathcal{O}(Y)$ to the unique vector field ξ_f satisfying:

$$\omega(\eta, \xi_f) = df(\eta),$$

for all vector fields η on Y. The vector field ξ_f is known as the Hamiltonian vector field associated with the function f. Then, we can define a bracket $\{\cdot, \cdot\}$ on $\mathcal{O}(Y)$ via:

$$\{f, g\} := \omega(\xi_f, \xi_g)$$

for any $f, g \in \mathcal{O}(Y)$. It follows from Theorem 1.2.7 of [CG10] that this indeed gives a Poisson algebra structure to $\mathcal{O}(Y)$.

Example 2.1.3. Another important example of Poisson algebras arises in the context of 'almost commutative' algebras. Let $\mathcal{A}$ be an associative (not necessarily commutative) algebra over $\mathbb{C}$ that has an increasing filtration $\{\mathcal{A}_i\}_{i\in\mathbb{N}}$. We define the associated graded algebra of $\mathcal{A}$ with respect to this filtration via $\mathrm{gr}(\mathcal{A}) := \oplus_{i\in\mathbb{N}}\mathcal{A}_i/\mathcal{A}_{i-1}$. Then, $\mathrm{gr}(\mathcal{A})$ is an associative algebra with a product induced by the one on $\mathcal{A}$. The algebra $\mathcal{A}$ is said to be almost commutative if the algebra $\mathrm{gr}(\mathcal{A})$ is commutative.

Given an almost commutative algebra $\mathcal{A}$ and its associated graded $A = \mathrm{gr}(\mathcal{A})$, the algebra A has a natural Poisson structure defined as follows. Define:

$$\{\cdot, \cdot\} : \mathcal{A}_i/\mathcal{A}_{i-1} \times \mathcal{A}_j/\mathcal{A}_{j-1} \longrightarrow \mathcal{A}_{i+j-1}/\mathcal{A}_{i+j-2}$$

$$\{a_i, a_j\} := \widetilde{a_i}\widetilde{a_j} - \widetilde{a_j}\widetilde{a_i},$$

where $\widetilde{a_k}$ denotes some lift of $a_k \in \mathcal{A}_k$ for $k = i, j$ to $\mathcal{A}$. The almost commutativity of $\mathcal{A}$ implies that this bracket is well-defined and that the map doesn't depend on the choice of the lift. Then, it is straightforward to verify that $\{\cdot, \cdot\}$ indeed defines a Poisson structure on A.

The most important example of a Poisson algebra that we will be dealing with lies in the intersection of the two examples above: Let $Y = T^*(X)$ for some smooth affine algebraic variety X. Then, Y is symplectic, and so by Example 2.1.2, we get a Poisson algebra structure on the ring of regular functions $\mathcal{O}(Y)$.

Next, let $\mathcal{D}(X)$ denote the ring of algebraic differential operators on X. This algebra has a filtration given by the order of the differential operator, and it turns out that the associated graded with respect to this filtration is exactly $\mathcal{O}(T^*(X)) = \mathcal{O}(Y)$. Hence, by Example 2.1.3, we get another Poisson algebra structure on $\mathcal{O}(Y)$. Fortunately, we have the following result:

Proposition 2.1.1. *[CG10, Theorem 1.3.10] The two Poisson structures on $\mathcal{O}(Y)$ described above coincide.*

Let V be a symplectic vector space and let W be a subspace of V. Let $W^\perp \subseteq V$ denote the annihilator of W with respect to the symplectic form on V.

Definition 2.1.3.

1. The subspace W is said to be isotropic if $W \subseteq W^\perp$.
2. The subspace W is said to be co-isotropic if $W^\perp \subseteq W$.
3. The subspace W is said to be Lagrangian if $W = W^\perp$.

These definitions can be generalized to arbitrary symplectic varieties:

Definition 2.1.4. Let M be a symplectic variety. A (possibly singular) subvariety Z of M is said to be isotropic (resp. co-isotropic, Lagrangian) if at any point p in the smooth locus Z, the tangent space T_pZ is an isotropic (resp. co-isotropic, Lagrangian) subspace of T_pM.

It is easy to see that the dimension of an isotropic subvariety Z is less than or equal to half the dimnesion of M, that of a co-isotropic subvariety is greater than or equal to half the dimension of M and the dimension of a Lagrangian subvariety is exactly half the dimension of M.

We'll be needing the following proposition about subvarieties of isotropic subvarieties:

Proposition 2.1.2. *[CG10, Proposition 1.3.30] Let M be a symplectic variety and let Z be an isotropic subvariety. Then, any subvariety of Z is itself isotropic.*

2.2 Hamiltonian reduction

Let M be a symplectic variety with symplectic form ω. Suppose a reductive algebraic group G acts on M while preserving the symplectic form, that is, $\omega(x, y) = \omega(gx, gy)$ for all $x, y \in T_m M$ and $g \in G$ for any point $m \in M$. The infinitesimal group action gives rise to a Lie algebra homomorphism ϕ from the Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$ to the space of vector fields on M.

Definition 2.2.1.

1. The G-action on M is said to be Hamiltonian if there exists a G-equivariant map $\mu : M \to \mathfrak{g}^*$ such that the pullback map $\mu^* : \mathfrak{g} \to \mathcal{O}(M)$ satisfies $\xi_{\mu^*(x)} = \phi(x)$ for all $x \in \mathfrak{g}$.
2. The maps μ and μ^* in the above situation are known as the moment and the co-moment maps respectively for the G-action on M.

The fact that ϕ is a Lie algebra homomorphism implies that μ^* is a Lie algebra homomorphism, where the Lie algebra structure on $\mathcal{O}(M)$ is given by the Poisson bracket.

Example 2.2.1. Suppose a group G acts on a variety X. Then, the induced G-action on the symplectic variety $T^*(X)$ is always Hamiltonian whose co-moment map can be described as follows: Given $x \in \mathfrak{g}$, let v_x denote the vector field on $T^*(X)$ obtained due to the infinitesimal G-action. Then, we have the co-moment map:

$$\mu^* : \mathfrak{g} \longrightarrow \mathcal{O}(T^*(X))$$

$$x \mapsto \lambda(v_x),$$

where λ is the canonical 1-form on $T^*(X)$.

Definition 2.2.2. Let $\mathbb{O}$ denote a co-adjoint G-orbit in $\mathfrak{g}^*$. Given a Hamiltonian G-action on a symplectic variety M with moment map μ, we define the classical Hamiltonian reduction of M at the orbit $\mathbb{O}$ to be the categorical quotient $\mu^{-1}(\overline{\mathbb{O}})//G$.

We can also state an algebraic version of this definition. Given the co-moment map $\mu^* : \mathfrak{g} \to \mathcal{O}(M)$, extend it multiplicatively to get a map $\mu^* : \mathrm{Sym}(\mathfrak{g}) \to \mathcal{O}(M)$. Then, if $\mathcal{I}$ denotes the defining ideal of $\overline{\mathbb{O}}$ in $\mathrm{Sym}(\mathfrak{g})$, the classical Hamiltonian reduction of M at $\mathbb{O}$ is the variety $\mathrm{Spec}((\mathcal{O}(M)/\mu^*(\mathcal{I}))^G)$.

Remark 2.2.1. *The motivation for Hamiltonian reduction comes from the theory of integrable systems in classical mechanics. The idea is to model the configuration space of a given physical system as a symplectic variety, and then, use the technique of Hamiltonian reduction to get a symplectic variety of smaller dimension, which might aid in solving the relevant Hamilton's equations. See [Eti09, §2] for a more detailed description of the role of Hamiltonian reduction in classical mechanics.*

Now, we generalize the above notions to the non-commutative setting. Let $\mathcal{A}$ be an associative (not necessarily commutative) algebra. Suppose the Lie algebra $\mathfrak{g}$ acts on $\mathcal{A}$ by derivations, i.e., there is a Lie algebra map $\phi : \mathfrak{g} \to \mathrm{Der}(\mathcal{A})$, where $\mathrm{Der}(\mathcal{A})$ is the space of derivations on $\mathcal{A}$.

Definition 2.2.3.

1. The $\mathfrak{g}$-action on $\mathcal{A}$ is said to be Hamiltonian if there exists a map $\Theta : \mathcal{U}\mathfrak{g} \to \mathcal{A}$ such that for all $x \in \mathfrak{g}$ and $a \in \mathcal{A}$, we have the equality $\phi(x)(a) = [\Theta(x), a]$.

2. The map Θ above is known as the quantum co-moment map for the $\mathfrak{g}$-action on $\mathcal{A}$.

Example 2.2.2. Suppose the group G acts on a variety M and let $\mathcal{A} = \mathcal{D}(M)$ denote the algebra of differential operators on M. Then, there is a natural action of G on the algebra $\mathcal{A}$ and we can describe the quantum co-moment map as follows. The infinitesimal G-action results in a Lie algebra homomorphism $\Theta : \mathfrak{g} \to \{\text{Vector fields on } M\}$. This gives rise to an algebra homomorphism $\Theta : \mathcal{U}\mathfrak{g} \to \mathcal{D}(M)$, which is exactly the quantum co-moment map for the above action. Furthermore, this map Θ is a quantization of the classical co-moment map for the G-action on $T^*(M)$. More precisely, if $\mu^* : \mathrm{Sym}(\mathfrak{g}) \to \mathcal{O}(T^*(M))$ is the classical co-moment map corresponding to the G-action on $T^*(M)$, then $\mathrm{gr}(\Theta) = \mu^*$. (See [CG10, §1.4] for more details.)

Next, motivated by the algebraic definition of classical Hamiltonian reduction, we define the notion of quantum Hamiltonian reduction. Let $\mathcal{I} \subseteq \mathcal{U}\mathfrak{g}$ be a 2-sided ideal. Given a $\mathfrak{g}$-action on an algebra $\mathcal{A}$ and the corresponding quantum co-moment map Θ, define the left ideal $\mathcal{K} := \mathcal{A} \cdot \Theta(\mathcal{I})$ of $\mathcal{A}$. Then, it is a standard exercise to check that the ideal $\mathcal{K}^{\mathfrak{g}}$ is a two-sided ideal of the invariant subalgebra $\mathcal{A}^{\mathfrak{g}}$.

Definition 2.2.4. The quantum Hamiltonian reduction of the algebra $\mathcal{A}$ at the ideal $\mathcal{I}$ is defined as the associative algebra $\mathcal{A}^{\mathfrak{g}}/\mathcal{K}^{\mathfrak{g}} = (\mathcal{A}/\mathcal{K})^{\mathfrak{g}}$.

Example 2.2.3. Considering the same situation as in the previous example, we can consider the quantum Hamiltonian reduction of the algebra $\mathcal{A} = \mathcal{D}(\mathfrak{g})$ at the augmentation ideal $\mathcal{U}\mathfrak{g}^+ \subseteq \mathcal{U}\mathfrak{g}$. This is equal to the associative algebra $\Big(\mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g}) \cdot \Theta(\mathcal{U}\mathfrak{g}^+)\Big)^{\mathfrak{g}}$. When $\mathfrak{g}$ is reductive, this algebra is known to be isomorphic to the algebra $\mathcal{D}(\mathfrak{h})^W$, where $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra and W is the Weyl group. This is a consequence of Harish-Chandra's 'radial parts' homomorphism as stated in the introduction.

This result was generalized in [EG02] and [GG06] for the Lie algebra $\mathfrak{g} = \mathfrak{gl}_n$ as follows: There exists a one-parameter family of primitive ideals $\mathcal{J}_k \subseteq \mathcal{U}\mathfrak{g}$ for $k \in \mathbb{C}$ such that $\mathcal{J}_0 = \mathcal{U}\mathfrak{g}^+$ (see [Jos74].) Then, it was shown in these papers (using the construction that we review here in §2.4) that the quantum Hamiltonian reduction of the algebra $\mathcal{D}(\mathfrak{g})$ at the ideal $\mathcal{J}_k$ is isomorphic to the spherical subalgebra eH_ke of the rational Cherednik algebra H_k of Type A with parameter $k \in \mathbb{C}$.

2.3 The symplectic Lie algebra

Let V be a symplectic vector space and let $\mathfrak{g} = \mathfrak{sp}(V)$ be the Lie algebra of linear endomorphisms of V that preserve the symplectic form. Let $G = Sp(V)$ denote the symplectic group, so that $\mathfrak{g} = \mathrm{Lie}(G)$. In this section, we recall some explicit details about the root system associated with the Lie algebra $\mathfrak{g}$ and the nilpotent G-orbits in $\mathfrak{g}$ under the adjoint action.

2.3.1 Root system of Type C

For computations in subsequent sections, we fix some coordinates. We identify the symplectic vector space V with the space $\mathbb{C}^{2n}$, which has the standard symplectic form given by the following $2n \times 2n$ skew-symmetric matrix:

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

The subspace $L \subseteq V = \mathbb{C}^{2n}$, consisting of vectors whose last n coordinates are zero, is a Lagrangian subspace of V under the above symplectic form and the dual vector space L^* can be identified with the space of vectors whose first n coordinates are zero.

The Lie algebra $\mathfrak{sp}(V) = \mathfrak{sp}_{2n}$ is the space of endomorphisms preserving this form. Then, $\mathfrak{sp}(V)$ can be viewed as the subspace of $\mathfrak{gl}(V) = \mathfrak{gl}_{2n}$ consisting of block matrices of the form:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A = -D^T, B = B^T$ and $C = C^T$. The subspace $\mathfrak{h}$ consisting of diagonal matrices is a Cartan subalgera of $\mathfrak{sp}_{2n}$. Identifying $\mathfrak{h} \simeq \mathfrak{h}^*$ using the trace form, we fix an orthonormal basis $r_1, r_2, \ldots, r_n$ of $\mathfrak{h}^*$ given by $r_i = (E_{i,i} - E_{i+n,i+n})/\sqrt{2}$, where $E_{i,j} \in \mathfrak{gl}_{2n}$ is the elementary matrix whose only non-zero entry is in the i^{th} row of the j^{th} column and is equal to 1.

Then, we have a choice of a root system R given by:

$$R := \left\{ \pm \frac{(r_i + r_j)}{\sqrt{2}}, \pm \frac{(r_i - r_j)}{\sqrt{2}}, \pm\sqrt{2}r_i : 1 \leq i < j \leq n \right\}.$$

Here, the long roots are given the vectors $\pm\sqrt{2}r_i$, whereas the rest are all short roots. The set of positive roots $R^+ \subseteq R$ is obtained by replacing all '$\pm$'-signs by '$+$' in the above defintion. Let $\{e_\alpha\}_{\alpha \in R}$ denote the set of root vectors in $\mathfrak{h}$ that form a Cartan-Weyl basis of the Lie algebra $\mathfrak{g}$ chosen so that $(e_\alpha, e_{-\alpha}) = 1$, where $(.,.)$ is the trace form. Then, we can express the e_α's explicitly in terms of elementary matrices $E_{i,j}$ as follows:

$$\alpha = \frac{(r_i + r_j)}{\sqrt{2}} \implies e_\alpha = e_{-\alpha}^T = \frac{E_{i,j+n} + E_{j,i+n}}{\sqrt{2}}$$

$$\alpha = \frac{(r_i - r_j)}{\sqrt{2}} \implies e_\alpha = e_{-\alpha}^T = \frac{E_{i,j} - E_{j+n,i+n}}{\sqrt{2}}$$

$$\alpha = \sqrt{2}r_i \implies e_\alpha = e_{-\alpha}^T = E_{i,i+n}.$$

2.3.2 Nilpotent conjugacy classes

The nilpotent cone $\mathcal{N} \subseteq \mathfrak{sp}(V)$ is invariant under the action of the group G and is the union of finitely many G-orbits. More precisely:

Proposition 2.3.1. *[CM93, Theorem 5.1.3] The nilpotent conjugacy classes of $\mathfrak{sp}(V)$ are parametrized by the set P_n of all partitions of the integer $2n$ such that each odd part occurs with even multiplicity. (Here, $2n = \dim(V)$.)*

Given a partition $\lambda \in P_n$, we can construct a nilpotent conjugacy class of $\mathfrak{sp}(V)$ as follows. We'll be using the same notation as used in the previous section. For any even part

$2k$ that occurs in λ, we associate the $2k \times 2k$ symplectic matrix:

$$\begin{pmatrix} I & \epsilon \\ 0 & -I \end{pmatrix},$$

where I is the $k \times k$ identity and ϵ is the $k \times k$ elementary matrix $E_{k,k}$. Next, for any odd k, to any (k, k) occurring in λ, we associate the $2k \times 2k$ symplectic matrix:

$$\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

Then, a representative in the nilpotent conjugacy class represented by the partition λ is obtained by placing the above matrices as diagonal blocks for each part occurring in λ.

2.4 Universal Harish-Chandra homomorphism

We recall the construction of the universal Harish-Chandra homomorphism from [EG02]. In this section, let $\mathfrak{g}$ be an arbitrary reductive Lie algebra. Fix a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ and let W be the Weyl group. Let $\mathfrak{g}^{rs}$ denote the subset of regular semisimple elements of $\mathfrak{g}$ and let $\mathfrak{h}^{reg} = \mathfrak{h} \cap \mathfrak{g}^{rs}$. Then, $\mathfrak{h}^{reg}$ is the set of regular elements of $\mathfrak{h}$, which is equal to the complement of the root hyperplanes. Next, inside the universal enveloping algebra $\mathcal{U}\mathfrak{g}$, consider the space $(\mathcal{U}\mathfrak{g})^{\mathrm{ad}(\mathfrak{h})}$ of $\mathrm{ad}(\mathfrak{h})$-invariants. Then, $(\mathcal{U}\mathfrak{g})^{\mathrm{ad}(\mathfrak{h})} \cdot \mathfrak{h}$ is a two-sided ideal of the algebra $(\mathcal{U}\mathfrak{g})^{\mathrm{ad}(\mathfrak{h})}$, and so, we can define the quotient algebra $(\mathcal{U}\mathfrak{g})_{\mathfrak{h}} := (\mathcal{U}\mathfrak{g})^{\mathrm{ad}(\mathfrak{h})}/((\mathcal{U}\mathfrak{g})^{\mathrm{ad}(\mathfrak{h})} \cdot \mathfrak{h})$.

By Proposition 6.1 of [EG02], and the paragraph following its proof, there exists a canonical algebra isomorphism:

$$\Psi : \mathcal{D}(\mathfrak{g}^{reg})^{\mathfrak{g}} \longrightarrow \left(\mathcal{D}(\mathfrak{h}^{reg}) \otimes (\mathcal{U}\mathfrak{g})_{\mathfrak{h}}\right)^{W}.$$

Next, fix a $\mathcal{U}\mathfrak{g}$-module $\mathfrak{V}$ and let $\mathfrak{V}\langle 0 \rangle$ denote its zero weight space. Then, by definition, this space $\mathfrak{V}\langle 0\rangle$ is acted upon trivially by $\mathfrak{h}$, and is thus stable under the action of $(\mathcal{U}\mathfrak{g})^{\mathrm{ad}(\mathfrak{h})}$. Hence, we have an action of $(\mathcal{U}\mathfrak{g})_{\mathfrak{h}}$ on $\mathfrak{V}\langle 0\rangle$, giving an algebra homomorphism $\chi : (\mathcal{U}\mathfrak{g})_{\mathfrak{h}} \to \mathrm{End}_{\mathbb{C}}(\mathfrak{V}\langle 0\rangle)$. In particular, if $V\langle 0\rangle$ is one-dimensional, we get a homomorphism $\chi : (\mathcal{U}\mathfrak{g})_{\mathfrak{h}} \to \mathbb{C}$. So, we can compose it with the above algebra isomorphism and restrict the map to $\mathcal{D}(\mathfrak{g})^{\mathfrak{g}} \subseteq \mathcal{D}(\mathfrak{g}^{reg})^{\mathfrak{g}}$ to get a map:

$$\Psi_{\mathfrak{V}} := \chi \circ \Psi : \mathcal{D}(\mathfrak{g})^{\mathfrak{g}} \longrightarrow \mathcal{D}(\mathfrak{h}^{reg})^{W}.$$

This is the Harish-Chandra homomorphism associated with the representation $\mathfrak{V}$.

Remark 2.4.1. *When $\mathfrak{V}$ is the trivial representation of $\mathfrak{g}$, the map $\Psi_{\mathfrak{V}}$ is exactly the homomorphism studied by Harish-Chandra in [HC64]. In particular, the image of the map $\Psi_{\mathfrak{V}}$ is exactly $\mathcal{D}(\mathfrak{h})^{W} \subseteq \mathcal{D}(\mathfrak{h}^{reg})^{W}$ and the kernel is equal to the ideal $(\mathcal{D}(\mathfrak{g}) \cdot \mathrm{ad}(\mathfrak{g}))^{\mathfrak{g}} \subseteq \mathcal{D}(\mathfrak{g})^{\mathfrak{g}}$.*

Let $\mathrm{Ann}(\mathfrak{V}) \subseteq \mathcal{U}\mathfrak{g}$ be the annihilator of the representation $\mathfrak{V}$ and consider the ideal $(\mathcal{D}(\mathfrak{g}) \cdot \mathrm{ad}(\mathrm{Ann}(\mathfrak{V})))^{\mathfrak{g}}$, which is a two-sided ideal inside $\mathcal{D}(\mathfrak{g})^{\mathfrak{g}}$. Then, it follows from definitions and the proof of [EG02, Proposition 6.1] that the kernel of the homomorphism $\Psi_{\mathfrak{V}}$ contains the ideal $(\mathcal{D}(\mathfrak{g}) \cdot \mathrm{ad}(\mathrm{Ann}(\mathfrak{V})))^{\mathfrak{g}}$.

There also exists a group-theoretic version of the 'universal' Harish-Chandra homomorphism. For this, let G be an algebraic group corresponding to the reductive Lie algebra $\mathfrak{g}$ and fix a maximal torus $H \subseteq G$ such that $\mathrm{Lie}(H) = \mathfrak{h}$. Let $G^{reg} \subseteq G$ denote the regular, semisimple locus inside G and let $H^{reg} = G^{reg} \cap H$. Then, we have the following proposition about the differential operators on G^{reg} and H^{reg}:

Proposition 2.4.2. *There exists a cononical algebra isomorphism:*

$$\mathcal{D}(G^{reg})^{\mathfrak{g}} \longrightarrow (\mathcal{D}(H^{reg}) \otimes (\mathcal{U}\mathfrak{g})_{\mathfrak{h}})^{W}.$$

The proof of this proposition is obtained by mimicking the proof of Proposition 6.1

of [EG02] line-by-line. With this, we can repeat the construction above so that for every representation $\mathfrak{V}$ of $\mathfrak{g}$ having a one-dimensional zero weight space, we get an algebra homomorphism:

$$\Psi_{\mathfrak{V}}^{trig} : \mathcal{D}(G)^{\mathfrak{g}} \longrightarrow \mathcal{D}(H^{reg})^W,$$

such that the ideal $(\mathcal{D}(\mathfrak{g}) \cdot \mathrm{ad}(\mathrm{Ann}(\mathfrak{V})))^{\mathfrak{g}}$ lies in the kernel of $\Psi_{\mathfrak{V}}^{trig}$. (The reason for the 'trig' superscript will become clear in §5.2.)

2.5 Cherednik algebras of Type C

Fix a Cartan sublagebra $\mathfrak{h} \subseteq \mathfrak{g} = \mathfrak{sp}(V)$ and a root system $R \subseteq \mathfrak{h}^*$. The space $\mathfrak{h}$ (and hence $\mathfrak{h}^*$) has an action of the Weyl group $W = (\mathbb{Z}/(2))^n \rtimes S_n$. For each $\alpha \in R$, let $s_\alpha \in W$ denote the reflection of $\mathfrak{h}$ relative to the root α. Fix a W-invariant function $c : R \to \mathbb{C}$. For root systems of Type C, there are exactly two W-orbits in R given by the set of all long roots and the set of all short roots. Hence, such a W-invariant function c can be viewed as a pair of complex numbers $c = (c_{long}, c_{short}) \in \mathbb{C}^2$.

We recall from [EG02] the definition of the rational Cherednik algebra H_c of Type C, which is the one generated by the algebras $\mathrm{Sym}(\mathfrak{h}), \mathbb{C}[\mathfrak{h}](\simeq \mathrm{Sym}(\mathfrak{h}^*))$ and the group algebra $\mathbb{C}[W]$ with defining relations given by:

$$wxw^{-1} = w(x), wyw^{-1} = w(y),$$

$$[y, x] = \langle x, y\rangle - \frac{1}{2} \sum_{\alpha \in R} c(\alpha)\langle \alpha, y\rangle\langle x, \alpha^\vee\rangle \cdot s_\alpha,$$

for all $w \in W$, $x \in \mathfrak{h}^*$ and $y \in \mathfrak{h}$. This algebra arises as a rational degeneration of the double affine Hecke algebra introduced and studied by Cherednik in [Che05].

Let $e = \frac{1}{|W|} \sum_{w \in W} w$ be the averaging idempotent in $\mathbb{C}[W]$. Then, we can consider the spherical subalgebra $eH_ce \subseteq H_c$, which will be our main object of interest. Next, we recall another description of the algebra eH_ce given in [EG02]. Define the rational Calogero-Moser operator L_c (also known as the Olshanetsky-Perelomov operator), which is a differential operator on the space $\mathfrak{h}^{reg}$, as follows (see [OP83]):

$$L_c := \Delta_{\mathfrak{h}} - \frac{1}{2}\sum_{\alpha \in R} \frac{c(\alpha)(c(\alpha)+1)}{\alpha^2} \cdot (\alpha, \alpha),$$

where $\Delta_{\mathfrak{h}}$ is the Laplacian operator on the Cartan subalgebra $\mathfrak{h}$. Let $\mathcal{D}(\mathfrak{h}^{reg})_-$ denote the subalgebra of $\mathcal{D}(\mathfrak{h}^{reg})$ spanned by differential operators $D \in \mathcal{D}(\mathfrak{h}^{reg})$ such that $degree(D) + order(D) \leq 0$. Let $\mathcal{C}_c$ denote the centralizer of the operator L_c in $\mathcal{D}(\mathfrak{h}^{reg})_-^W$. Finally, consider the subalgebra $\mathcal{B}_c$ of $\mathcal{D}(\mathfrak{h}^{reg})$ generated by $\mathcal{C}_c$ and $\mathbb{C}[\mathfrak{h}]^W$, the algebra of W-invariant polynomial functions on $\mathfrak{h}$.

Theorem 2.5.1. *[EG02, Proposition 4.9] We have an embedding (known as the Dunkl embedding) $\Theta : eH_ce \hookrightarrow \mathcal{D}(\mathfrak{h}^{reg})^W$ such that $\Theta(\mathbb{C}[\mathfrak{h}]^W) = \mathbb{C}[\mathfrak{h}]^W$ and $\Theta(\mathrm{Sym}(\mathfrak{h})^W) = \mathcal{C}_c$. Moreover, $\Theta(\Delta_{\mathfrak{h}}) = L_c$. Furthermore, Θ induces an isomorphism of algebras $eH_ce \simeq \mathcal{B}_c$.*

We give a description of the map Θ: We first describe the construction of a map $\widetilde{\Theta} : H_c \to \mathcal{D}(\mathfrak{h}^{reg})\#\mathbb{C}[W]$. Consider the subalgebra $\mathcal{S}$ of H_c generated by the subalgebras $\mathrm{Sym}(\mathfrak{h})$ and $\mathbb{C}[W]$ and let *triv* denote the trivial representation of the algebra $\mathcal{S}$. Then, we have the induced H_c-module $Ind_{\mathcal{S}}^{H_c}(triv)$ and it can be shown that the underlying vector space of this module can be identified with the space $\mathbb{C}[\mathfrak{h}]$. Then, the map $\widetilde{\Theta}$ is defined via the action of the algebra H_c on this module. We can describe this action explicitly via:

$$w \mapsto w, x \mapsto x$$

$$y \mapsto D_y := \frac{\partial}{\partial y} + \frac{1}{2}\sum_{\alpha \in R} c_\alpha \cdot \frac{\langle \alpha, y \rangle}{\alpha} \cdot (s_\alpha - 1),$$

for all $w \in W$, $x \in \mathfrak{h}^*$ and $y \in \mathfrak{h}$. The differential operators D_y are known as the Dunkl operators, first defined in [Dun89]. Then, the map Θ is obtained by conjugating the map $\widetilde{\Theta}$

by the element $\delta_c = \prod_{\alpha \in R} \alpha^{c_\alpha/2}$ and restricting it to the spherical subalgebra $eH_c e \subseteq H_c$. (See [EG02, §4] for more details.)

The algebra H_c has a filtration such that all the elements of W and the generators of $\mathrm{Sym}(\mathfrak{h}^*)$ have degree zero, and the generators of $\mathrm{Sym}(\mathfrak{h})$ have degree one. This induces a filtration on the spherical subalgebra $eH_c e$. The algebra $\mathcal{D}(\mathfrak{h}^{reg})$ has a filtration given by the order of differential operators. Then, we have the following PBW theorem for Cherednik algebras:

Proposition 2.5.2. *[EG02, Corollary 4.4, Proposition 4.9] We have an isomorphism of vector spaces:*

$$H_c \simeq \mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}[W].$$

Furthermore, with the filtration defined above, we have an isomorphism of commutative algebras:

$$\mathrm{gr}(eH_c e) \simeq \mathbb{C}[\mathfrak{h} \times \mathfrak{h}]^W.$$

Moreover, under this isomorphism, the associated graded version of the Dunkl embedding $\mathrm{gr}(\Theta) : \mathbb{C}[\mathfrak{h} \times \mathfrak{h}]^W \simeq \mathrm{gr}(eH_c e) \to \mathrm{gr}(\mathcal{D}(\mathfrak{h}^{reg})) = \mathbb{C}[\mathfrak{h}^{reg} \times \mathfrak{h}^*]^W$ *is exactly the algebra homomorhpism induced by the natural embedding* $\mathfrak{h}^{reg} \hookrightarrow \mathfrak{h}$.

The rational Cherednik algebra H_c has a very rich representation theory (see, for example, the works of Berest, Etingof and Ginzburg [BEG03a, BEG03b] and Ginzburg, Guay, Opdam and Rouquier [GGOR03].) By [EG02, Theorem 1.7], the algebra H_c is Morita equivalent to the subalgebra $eH_c e$, and so, the representation theories of these two algebras coincide.

Because of the PBW theorem above, we can consider a Category $\mathcal{O}$ of H_c-representations in analogy with the Category $\mathcal{O}$ of a semisimple Lie algebra $\mathfrak{g}$ defined by [BGfGf76]: We define the Category $\mathcal{O}$ associated with the Cherednik algebra H_c to be the full subcategory of H_c-mod consisting of all finitely generated H_c-modules having a locally finite action of the subalgebra $\mathrm{Sym}(\mathfrak{h}) \subseteq H_c$. Then, just like the [BGfGf76] case, the category $\mathcal{O}$ breaks down into a direct sum of blocks (see [Dix96]):

$$\mathcal{O} = \bigoplus_{\lambda \in \mathfrak{h}^*//W} \mathcal{O}_\lambda(H_c).$$

Here, $\mathcal{O}_\lambda(H_c)$ is the full subcategory of $\mathcal{O}$ consisting of modules M such that for any $P \in \mathrm{Sym}(\mathfrak{h})^W$, the action of $P - P(\lambda)$ on M is locally nilpotent. We'll be interested in the block $\mathcal{O}_0(H_c)$ corresponding to $\lambda = 0$, which we will refer to as simply $\mathcal{O}(H_c)$ by abuse of notation.

2.5.1 Trigonometric Cherednik algebras

Keeping the notation from the previous section, fix a maximal torus $H \subseteq G = Sp(V)$ such that $\mathrm{Lie}(H) = \mathfrak{h}$. Having fixed a root system R, let Q denote the root lattice and let P denote the weight lattice. The Weyl group has an action on both of these lattices, and we define the extended affine Weyl group $W^e := P \rtimes W$. Then, we have that the group algebra $\mathbb{C}[W^e]$ is isomorphic to the algebra $\mathbb{C}[H] \rtimes \mathbb{C}[W]$. Let C denote the fundamental alcove for the action of the Weyl group, i.e.:

$$C := \{\lambda \in \mathfrak{h} : \lambda(\alpha) > 0 \text{ for all simple roots } \alpha\}.$$

Let $\Omega = \{w \in W^e : w(C) = C\}$.

We recall from [Che05, Opd00] the definition of the trigonometric Cherednik algebra H_c^{trig} of Type C, as the one generated by the algebra $\mathrm{Sym}(\mathfrak{h})$ and the group algebra $\mathbb{C}[W^e]$ with defining relations given by:

$$s_\alpha \cdot y - s_\alpha(y) \cdot w = -c_\alpha \langle \alpha, y \rangle,$$

$$\pi \cdot y = \pi(y) \cdot \pi,$$

for all $\alpha \in R$, $y \in \mathfrak{h}$ and $\pi \in \Omega$. The algebra H_c^{trig}, also known as the degenerate double affine Hecke algebra, arises at an intermediate stage as the double affine Hecke algebra $\mathcal{H}_c$ degenerates to the rational Cherednik algebra H_c. That is, the algebra $\mathcal{H}_c$ contains the group algebras of the groups P and $P^\vee$ (the dual weight lattice) as subalgebras and the algebra H_c^{trig} is effectively obtained from $\mathcal{H}_c$ by replacing the subalgebra $\mathbb{C}[P^\vee]$ by the algebra $\mathrm{Sym}(\mathfrak{h})$. If the subalgebra $\mathbb{C}[P]$ is replaced by the algebra $\mathrm{Sym}(\mathfrak{h}^*)$ too, we obtain the rational Cherednik algebra H_c.

Let $e = \frac{1}{|W|}\sum_{w\in W} w$ be the averaging idempotent in H_c^{trig} and let $eH_c^{trig}e$ denote the spherical subalgebra. As in the rational case, we have an embedding of the algebra $eH_c^{trig}e$ in the algebra of differential operators $\mathcal{D}(H^{reg})$. To see this, we recall the trigonometric Calogero-Moser operator L_c^{trig} on the space H^{reg}, which is defined as follows (see [Sut71]):

$$L_c^{trig} := \Delta_H - \frac{1}{2}\sum_{\alpha\in R}\frac{c(\alpha)(c(\alpha)+1)}{\sin^2(\alpha)}\cdot(\alpha,\alpha),$$

where Δ_H is the Laplacian operator on the group H. Next, as in the rational case, we can define the trigonometric Dunkl-Heckman operators (see [Hec97]):

$$D_y^{trig} := \frac{\partial}{\partial y} + \frac{1}{2}\sum_{\alpha\in R} c_\alpha \cdot \frac{\langle\alpha, y\rangle}{1-e^{-\alpha}}\cdot(s_\alpha - 1),$$

for a given $y \in \mathfrak{h}$. Let $\mathfrak{S}$ denote the subalgebra of $\mathcal{D}(H^{reg})$ generated the operators D_y^{trig} for all $y \in \mathfrak{h}$.

Theorem 2.5.3. *[Opd00, Thereom 3.7] There exists an embedding $\Sigma : H_c^{trig} \hookrightarrow \mathcal{D}(H^{reg})$ such that:*

$$\Sigma(\lambda) = \lambda,$$

$$\Sigma(y) = D_y^{trig}$$

for all $\lambda \in P \subseteq W^e$ and $y \in \mathfrak{h}$. Moreover, $\Sigma(\Delta_H) = L_c^{trig}$. In particular, the Dunkl-Heckman *operators commute with each other as well as with the operator* L_c^{trig}.

We define a filtration on H_c^{trig} such that all the elements of W^e have degree zero, and the generators of $\mathrm{Sym}(\mathfrak{h})$ have degree one. This induces a filtration on the spherical subalgebra $eH_c^{trig}e$. The algebra $\mathcal{D}(H^{reg})$ has a filtration given by the order of differential operators. Then, we have the following theorem:

Proposition 2.5.4. *[Eti17, Theorem 2.17, Theorem 2.29] We have an isomorphism of vector spaces:*

$$H_c^{trig} \simeq \mathbb{C}[H] \otimes \mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}[W].$$

Furthermore, with the filtration defined above, we have an isomorphism of commutative algebras:

$$\mathrm{gr}(eH_c^{trig}e) \simeq \mathbb{C}[H \times \mathfrak{h}]^W.$$

Moreover, under the above isomorphism, the associated graded map $\mathrm{gr}(\Sigma) : \mathbb{C}[H \times \mathfrak{h}]^W \simeq \mathrm{gr}(eH_c^{trig}e) \to \mathrm{gr}(\mathcal{D}(H^{reg})) = \mathbb{C}[H^{reg} \times \mathfrak{h}]^W$ *is exactly the algebra homomorphism induced by the natural embedding* $H^{reg} \hookrightarrow H$.

We end this section with the definition of category $\mathcal{O}$ for the spherical subalgebra $eH_c^{trig}e$. Note that the algebra $eH_c^{trig}e$ contains a subalgebra isomorphic to $\mathrm{Sym}(\mathfrak{h})^W$ and let $\mathrm{Sym}(\mathfrak{h})_+^W$ denote the augmentation ideal of this algebra.

Definition 2.5.1. The category $\mathcal{O}(eH_c^{trig}e)$ is defined as the full subcategory of the category of finitely generated $eH_c^{trig}e$-modules whose objects are $eH_c^{trig}e$-modules M that have a locally nilpotent action of the subalgebra $\mathrm{Sym}(\mathfrak{h})_+^W$.

CHAPTER 3

THE NILPOTENT SUBSCHEME OF THE ALMOST COMMUTING SCHEME

In this chapter, we consider the reduced subscheme X^{nil} of the almost commuting scheme X defined via:

$$X^{nil} := \{(x, y, i) \in \mathfrak{g} \times \mathfrak{g} \times V : [x, y] + i^2 = 0 \text{ and } y \text{ is nilpotent}\}.$$

In §3.1, we'll prove that this is a Lagrangian subscheme of $\mathfrak{g} \times \mathfrak{g} \times V$. We will then use this result to provide a new proof of Theorem 1.1.1. In §3.2, we compute the irreducible components of X^{nil}.

3.1 Lagrangian subscheme

To state the precise result, we first describe a symplectic structure on $\mathcal{X} := \mathfrak{g} \times \mathfrak{g} \times V$, which we define as $\omega = \omega_1 + \omega_2$. Here, ω_1 is the symplectic form on $\mathfrak{g} \times \mathfrak{g}$ obtained by identifying it with $T^*(\mathfrak{g})$ using the trace form on $\mathfrak{g}$ and ω_2 is the form on the symplectic vector space V. Next, on the scheme $\mathcal{M} := \mathfrak{gl}(V) \times \mathfrak{gl}(V) \times V \times V^*$, we have a symplectic form ω' obtained by identifying it with $T^*(\mathfrak{gl}(V) \times V)$ using the trace form on $\mathfrak{gl}(V)$.

Theorem 3.1.1. *The scheme X^{nil} is a Lagrangian complete intersection in $\mathcal{X}$.*

Proof. The scheme X^{nil} is a closed subscheme of $\mathcal{X}$ defined using $\dim(\mathfrak{g}) + \frac{1}{2}\dim(V)$ equations. The formula $[x, y]+i^2 = 0$ gives $\dim(\mathfrak{g})$ of these equations, whereas the other $\frac{1}{2}\dim(V)$ equations follow from the nilpotence condition on y. So, $\dim(X^{nil}) \geq \dim(\mathcal{X}) - (\dim(\mathfrak{g}) + \frac{1}{2}\dim(V)) = \dim(\mathfrak{g}) + \frac{1}{2}\dim(V) = \frac{1}{2}\dim(\mathcal{X})$. Therefore, to prove the theorem, it suffices to show that X^{nil} is isotropic. For this, consider the embedding:

$$\Phi : \mathcal{X} = \mathfrak{g} \times \mathfrak{g} \times V \longrightarrow \mathfrak{gl}(V) \times \mathfrak{gl}(V) \times V \times V^* = \mathcal{M}$$

$$(x, y, i) \mapsto (x, y, i_1, i_2),$$

where $i_1 = i/2$ and i_2 is the symplectic dual of i in V. (That is, i_2 is the image of i in V^* under the identification $V \simeq V^*$ using the symplectic form.)

Recall from [GG06] the scheme of almost commuting matrices $M \subseteq \mathcal{M}$:

$$M := \{(x, y, i, j) \in \mathfrak{gl}(V) \times \mathfrak{gl}(V) \times V \times V^* : [x, y] + ij = 0\}.$$

Also defined in [GG06] was the closed subscheme M^{nil} of M obtained by stipulating y to be nilpotent. Then, under the map Φ, we have $\Phi(X) \subseteq M$ and $\Phi(X^{nil}) \subseteq M^{nil}$.

We claim that with the symplectic forms ω and ω' defined above, the map Φ is a symplectic embedding. To see this, we first observe that we can express the form ω' as a sum $\omega'_1 + \omega'_2$, where ω'_1 is the symplectic form on $\mathfrak{gl}(V) \times \mathfrak{gl}(V)$ obtained by identifying it with $T^*(\mathfrak{gl}(V))$ and ω'_2 is the symplectic form on $V \times V^*$ obtained by identifying it with $T^*(V)$. Then, it is clear that $\omega_1 = \Phi^*(\omega'_1)$. Next, if i, j are two vectors in V, then we have:

$$\omega'_2((i_1, j_1), (i_2, j_2)) = j_1(i_2) - j_2(i_1) = \frac{1}{2}\omega_2(i, j) - \frac{1}{2}\omega_2(j, i) = \omega_2(i, j),$$

which shows that Φ preserves the symplectic structure.

By [GG06, Theorem 1.1.4], we know that that M^{nil} is a Lagrangian subscheme of $\mathcal{M}$. In particular, it is isotropic. Therefore, by Proposition 2.1.2, we get that $\Phi(X^{nil})$ is an isotropic subscheme of $\mathcal{M}$, proving that X^{nil} is an isotropic subscheme of $\mathcal{X}$. □

Corollary 3.1.2. *The scheme X is a complete intersection of dimension* $\dim(\mathfrak{g}) + \dim(V)$.

Proof. The scheme X^{nil} is obtained from X by imposing exactly $n = \frac{1}{2}\dim(V)$ equations, that come from imposing the nilpotence condition on y. Hence, as $\dim(X^{nil}) = \frac{1}{2}\dim(\mathcal{X}) = \dim(\mathfrak{g}) + \frac{1}{2}\dim(V)$, we must have that $\dim(X) \leq \dim(X^{nil}) + \frac{1}{2}\dim(V) = \dim(\mathfrak{g}) + \dim(V)$. But, the scheme X is obtained from $\mathcal{X}$ by imposing $\dim(\mathfrak{g})$ equations, and so, $\dim(X) \geq \dim(\mathcal{X}) - \dim(\mathfrak{g}) = \dim(\mathfrak{g}) + \dim(V)$. Therefore, X is a complete intersection of dimension $\dim(\mathfrak{g}) + \dim(V)$. □

In fact, we can generalize Theorem 3.1.1 as follows. Let $\mathfrak{h} \subseteq \mathfrak{g}$ denote a Cartan subalgebra of $\mathfrak{g}$ and let W be the Weyl group. Consider the composition map $\phi : \mathfrak{g} \to \mathfrak{g}//G \to \mathfrak{h}//W$, where the first map is the categorical quotient map and the second one is the Chevalley restriction isomorphism. Then, we can consider the morphism:

$$\pi : X \longrightarrow \mathfrak{h}//W,$$

that sends a triple (x, y, i) to $\phi(y)$. It is clear that $X^{nil} = \pi^{-1}(\{0\})$.

Proposition 3.1.3. *All the fibers of the map π are Lagrangian subschemes of $\mathcal{X}$ and have dimension* $\dim(\mathfrak{g}) + \frac{1}{2}\dim(V)$*.*

Proof. For any $x \in \mathfrak{h}//W$, we have the dimension inequality:

$$\dim(\pi^{-1}(\{x\})) \geq \dim(X) - \dim(\mathfrak{h}//W) = \dim(\mathfrak{g}) + \frac{1}{2}\dim(V) = \frac{1}{2}\dim(\mathcal{X}).$$

Therefore, to show that $\pi^{-1}(\{x\})$ is Lagrangian, it suffices to prove that it is isotropic.

We consider the symplectic embedding $\Phi : \mathcal{X} \to \mathcal{M}$ defined in the proof of Theorem 3.1.1. By Corollary 2.3.4 of [GG06], the image of $\pi^{-1}(\{x\})$ in $\mathcal{M}$ lies inside a Lagrangian subscheme of $\mathcal{M}$. As a result, we conclude that the image of $\pi^{-1}(\{x\})$ must be an isotropic subscheme of $\mathcal{M}$, showing that $\pi^{-1}(\{x\})$ must itself be isotropic. □

Remark 3.1.4. *In fact, by adapting the proof of Proposition 2.3.2 of [GG06], we can prove the following: Corresponding to the Hamiltonian G-action on the variety $\mathcal{X}$, we get a moment* *map $\mu : \mathcal{X} \to \mathfrak{g}^* \simeq \mathfrak{g}$ given by the formula $(x, y, i) \mapsto [x, y] + i^2$. Consider the map:*

$$\mu \times \pi : \mathcal{X} = \mathfrak{g} \times \mathfrak{g} \times V \longrightarrow \mathfrak{g} \times \mathfrak{h}//W,$$

that maps a triple (x, y, i) to the pair $([x, y] + i^2, \phi(y))$. Then, this map is a flat morphism. As a corollary of this fact, we also get that the moment map $\mu : \mathcal{X} \to \mathfrak{g}$ if flat.

With this, we are ready to prove Losev's theorem. Define the scheme:

$$X^{reg} := \{(x, y, i) \in X : y \text{ is regular, semisimple}\}.$$

In other words, $X^{reg} = \pi^{-1}(\mathfrak{h}^{reg})$, where $\mathfrak{h}^{reg}$ is the regular semisimple locus of $\mathfrak{h}$. By Lemma 2.9 of [Los21], the scheme X^{reg} is irreducible.

Theorem 3.1.5. *1. We have $\overline{X^{reg}} = X$. In particular, the scheme X is irreducible.*

2. The scheme X is a reduced, complete intersection of dimension $\dim(\mathfrak{g}) + \dim(V)$.

Proof. By Corollary 3.1.2, we already know that X is a complete intersection of $\dim(\mathfrak{g}) + \dim(V)$. Consider the big diagonal $\Delta = (\mathfrak{h} \setminus \mathfrak{h}^{reg})//W$, which is a closed subscheme of $\mathfrak{h}//W$ of codimension 1. Then, we have the equality:

$$X = \overline{X^{reg}} \cup \pi^{-1}(\Delta).$$

Since $\dim(\Delta) = \dim(\mathfrak{h}//W) - 1 = \frac{1}{2}\dim(V) - 1$, by Proposition 3.1.3, we have $\dim(\pi^{-1}(\Delta)) \leq \dim(\mathfrak{g}) + \dim(V) - 1$. However, as X is a complete intersection, any irreducible component must have dimension exactly $\dim(\mathfrak{g}) + \dim(V)$. Therefore, we conclude that $\overline{X^{reg}}$ must be the only irreducible component, proving that $X = \overline{X^{reg}}$.

Finally, we note that the action of the group G is generically free on X^{reg}, and so, X is generically reduced. Therefore, as X is a complete intersection, it must be Cohen-Macaulay, and thus, we conclude that X is reduced (see [CG10, Theorem 2.2.11].) □

3.2 Irreducible components of X^{nil}

Let P_n denote the set of all partitions of $2n$ where every odd part occurs an even number of times. Let $\mathcal{P}_n \subseteq P_n$ be the subset of those partitions where each part is even. For any $\lambda \in P_n$, let $\mathcal{N}_\lambda$ denote the corresponding nilpotent conjugacy class of $\mathfrak{g}$ and define:

$$X_\lambda := \{(x, y, i) \in X^{nil} : y \in \mathcal{N}_\lambda\}.$$

Then, it is clear that we have the following disjoint union:

$$X^{nil} = \coprod_{\lambda \in P_n} X_\lambda.$$

Theorem 3.2.1. *1. For each $\lambda \in \mathcal{P}_n$, we have $\dim(X_\lambda) = \dim(\mathfrak{g}) + \frac{1}{2}\dim(V)$.*

2. For each $\lambda \in P_n \setminus \mathcal{P}_n$, we have $\dim(X_\lambda) < \dim(\mathfrak{g}) + \frac{1}{2}\dim(V)$.

As X^{nil} has been shown to be a complete intersection of dimension $\dim(\mathfrak{g}) + \frac{1}{2}\dim(V)$, this theorem implies that the irreducible components of X^{nil} are given exactly by the closures of those X_λ for which $\lambda \in \mathcal{P}_n$, completing the proof of Theorem 1.1.2.

Proof. Let y be any nilpotent element in $\mathfrak{g}$. Let $\mathbb{O}$ denote the minimal orbit consisting of rank one elements in $\mathfrak{g}$. Define the reduced schemes: (Also defined in [Los21])

$$X_y = \{(x, i) \in \mathfrak{g} \times V : [x, y] + i^2 = 0\},$$

$$\underline{X_y} = \{(x, z) \in \mathfrak{g} \times \overline{\mathbb{O}} : [x, y] + z = 0\},$$

$$Y_y = \overline{\mathbb{O}} \cap \{[x, y] : x \in \mathfrak{g}\}.$$

We have maps:

$$\rho_1 : X_y \longrightarrow \underline{X_y}$$

$$(x, i) \mapsto (x, i^2),$$

and

$$\rho_2 : \underline{X_y} \longrightarrow Y_y$$

$$(x, z) \mapsto z.$$

The map ρ_1 is finite, with either one or two points in the fiber at any point depending on whether i^2 is zero or not, respectively. The map ρ_2 is an affine bundle map which has fibers of dimension equal to $\dim(\mathfrak{z}_{\mathfrak{g}}(y))$. Here, $\mathfrak{z}_{\mathfrak{g}}(y)$ is the centralizer of y in $\mathfrak{g}$. Thus, we get that $\dim(X_y) = \dim(\mathfrak{z}_{\mathfrak{g}}(y)) + \dim(Y_y)$. Further, if $y \in \mathcal{N}_\lambda$ for some $\lambda \in P_n$, we have that $\dim(X_\lambda) = \dim(X_y) + \dim(\mathcal{N}_\lambda) = \dim(\mathfrak{g}) + \dim(Y_y)$. So, in order to prove the theorem, we are required to show that $\dim(Y_y) = \frac{1}{2}\dim(V)$ for $\lambda \in \mathcal{P}_n$ and $\dim(Y_y) < \frac{1}{2}\dim(V)$ for $\lambda \notin \mathcal{P}_n$.

For a fixed nilpotent y, by Jacobson-Morozov theorem ([CG10, Theorem 3.7.1]), we can find an $\mathfrak{sl}_2$-triple (e, f, h) in $\mathfrak{g}$ with $e = y$. Identifying the subspace $\langle e, f, h\rangle \subseteq \mathfrak{g}$ with $\mathfrak{sl}_2$, we get an $\mathfrak{sl}_2$-action on the vector space V by restricting the action of $\mathfrak{g}$. By $\mathfrak{sl}_2$-theory, the element h acts semisimply on V with integer eigenvalues. Let $V = V_- \oplus V_0 \oplus V_+$ be the decomposition of V, such that V_-, V_0 and V_+ denote the spans of negative, zero and positive eigenspaces of h respectively.

Now, we consider the identification of $\mathfrak{sl}_2$-representations $\mathfrak{g} = \mathfrak{sp}(V) = \mathrm{Sym}^2(V)$. By Lemma 3.2.2 proved below, for any $v \in V$, we have:

$$v \in V_+ \iff v^2 = \mathrm{ad}_e(x) = \mathrm{ad}_y(x) = [y, x] \text{ for some } x \in \mathfrak{g}.$$

Hence, we conclude that $\dim(Y_y) = \dim(V_+)$. As there is a one-to-one correspondence between positive and negative eigenvectors of h, we have that $\dim(V_-) = \dim(V_+)$, and so, $\dim(Y_y) \leq \frac{1}{2}\dim(V)$. The equality holds exactly when $V_0 = 0$, that is, when h has no zero eigenvalues. Zero eigenvalues for h occur only in irreducible representations having odd dimension. Hence, the dimension inequality becomes an equality exactly when the space V decomposes into a sum of irreducibles each having even dimension. However, irreducible components of V correspond exactly to the Jordan blocks of $e(= y)$. Therefore, the dimension equality holds exactly when each Jordan block of y has even size, that is $y \in \mathcal{N}_\lambda$ for some $\lambda \in \mathcal{P}_n$, thus completing the proof. $\square$

Lemma 3.2.2. *Let V be a representation of $\mathfrak{sl}_2(\mathbb{C}) = \langle e, f, h \rangle$. Let $W = \mathrm{Sym}^2(V)$ with the action of $\mathfrak{sl}_2$ induced from the one on V. Then, for any given $v \in V$, we have that $v^2 = e \cdot w$ for some $w \in W$ if and only if v lies in the span of the positive eigenspaces of h in V.*

Proof. Suppose v lies in the span of the positive eigenspaces of h in V. Then, we must have that v^2 lies in the span of the positive eigenspaces of h in W, and thus, by $\mathfrak{sl}_2$-theory, we must have $v^2 = e \cdot w$ for some $w \in W$.

Now, we prove the converse. Henceforth, instead of working with $\mathrm{Sym}^2(V)$, we will work with $W = V \otimes V$, for notational convenience. As $V \otimes V$ contains $\mathrm{Sym}^2(V)$ as a proper subrepresentation, if $v^2 = e \cdot w$ for some $w \in \mathrm{Sym}^2(V)$, we must have that $v \otimes v = e \cdot \widetilde{w}$ for some $\widetilde{w} \in V \otimes V$.

Let $V = \oplus_i V_i$ be the decomposition of V into a direct sum of $\mathfrak{sl}_2$-representations. Then, we get a corresponding decomposition $v = \sum_i v_i$ such that $v_i \in V_i$ for all i. Then, as $v \otimes v = e \cdot w$, it is clear that there exist $w_i \in V_i \otimes V_i$ such that $v_i \otimes v_i = e \cdot w_i$ for each i. Therefore, without loss of generality, we can assume that V is irreducible.

Suppose $\dim(V) = n + 1$ for some $n \in \mathbb{Z}_{\geq 0}$. Then, there exists an h-eigenbasis $\{x_0, x_1, \ldots, x_n\}$ of V such that the actions of e and f are given by:

$$e \cdot x_i = x_{i+1}, f \cdot x_j = x_{j-1},$$

for all $0 \leq i \leq n - 1$ and $1 \leq j \leq n$.

Write $v = \sum_i c_i x_i$ for $c_i \in \mathbb{C}$. Then, we have $v \otimes v = \sum_{i,j} c_i c_j x_i \otimes x_j$. Pick the smallest k such that $c_k \neq 0$. Then, $c_k^2 x_k \otimes x_k$ is the summand in $v \otimes v$ having the strictly smallest eigenvalue for the h-action. Therefore, as we have that $v \otimes v = e \cdot w$ for some $w \in V \otimes V$, there must exist $w' \in V \otimes V$ such that $x_k \otimes x_k = e \cdot w'$. Hence, without loss of generality, we can assume that $v = x_k$ for some k.

So, we are reduced to showing that given $x_k \otimes x_k = e \cdot w$ for some $w \in V \otimes V$, we must have that h acts on x_k with a positive eigenvalue. For the sake of a contradiction, suppose h acts on x_k with a non-positive eigenvalue. Therefore, we must have that h acts on $x_k \otimes x_k$ with a non-positive eigenvalue. As $x_k \otimes x_k = e \cdot w$, we must have that h acts on w with a strictly negative eigenvalue. Let $V' \subseteq V$ be the subspace spanned by $\{x_0, x_1, \ldots, x_{n-1}\}$ (that is, all but the highest weight vector). Then, we must have that $w \in V' \otimes V' \subseteq V \otimes V$.

Consider the linear function defined via:

$$f : V \otimes V \longrightarrow \mathbb{C}$$

$$x_i \otimes x_j \mapsto (-1)^i.$$

We claim that $f(e \cdot x) = 0$ for all $x \in V' \otimes V'$. To see this, we note that $V' \otimes V'$ is spanned by vectors for the form $x_i \otimes x_j$ for $0 \leq i, j \leq n-1$. For such i and j, we have:

$$e \cdot (x_i \otimes x_j) = x_{i+1} \otimes x_j + x_i \otimes x_{j+i},$$

which makes it clear that $f(e \cdot (x_i \otimes x_j)) = 0$.

In particular, since $w \in V' \otimes V'$, we must have $f(e \cdot w) = 0$. This implies that $f(x_k \otimes x_k) = 0$. However, it follows from the definition of f that $f(x_k \otimes x_k) = (-1)^k \neq 0$. This gives a contradiction, and so, the eigenvalue corresponding to x_k must have been positive, completing the proof. $\square$

CHAPTER 4

HAMILTONIAN REDUCTION

4.1 Classical setting

In this section, we prove the isomorphism $X//\{\pm 1\} \simeq A$ (see Theorem 4.1.3). We will first define some notation to formulate the precise statement. All tensor products will be over $\mathbb{C}$, unless specified otherwise.

4.1.1 The defining ideal of the minimal orbit of $\mathfrak{sp}(V)$

Let ω denote the symplectic form on the vector space V. Owing to the natural action of G on the vector space V that preserves ω, we get a moment map:

$$\mu_1 : V \longrightarrow \mathfrak{g}^*,$$

that maps the element $v \in V$ to $v^2 \in \mathfrak{g} \simeq \mathfrak{g}^*$. Here, we use the identification $\mathrm{Sym}^2(V) = \mathfrak{g}$. We can dualize this map to get a co-moment map:

$$\theta_1 : \mathfrak{g} \longrightarrow \mathbb{C}[V].$$

By [CG10, Proposition 1.4.6], we have the following formula for this co-moment map: Any element $x \in \mathfrak{g}$ maps under θ_1 to the polynomial function on V given by:

$$v \mapsto \frac{1}{2}\omega(x \cdot v, v)$$

for all $v \in V$. In particular, the image of θ_1 in $\mathbb{C}[V]$ is exactly the vector space of polynomial functions on V having degree 2. We can extend the above map multiplicatively to get a map $\mathbb{C}[\mathfrak{g}] \simeq \mathrm{Sym}(\mathfrak{g}) \to \mathbb{C}[V]$, which we also call θ_1, whose image is exactly the subalgebra

$\mathbb{C}[V]_{even} \subseteq \mathbb{C}[V]$ of polynomials that have even total degree. Define $K := \ker(\theta_1) \subseteq \mathbb{C}[\mathfrak{g}]$. We give a more explicit description of this ideal K below.

Lemma 4.1.1. *Let $\overline{\mathbb{O}}$ denote the closure of the orbit $\mathbb{O}$ of rank one matrices in $\mathfrak{g}$, such that the scheme structure on $\overline{\mathbb{O}}$ is given by the reduced structure on it. Then, the radical ideal in $\mathbb{C}[\mathfrak{g}]$ that defines the scheme $\overline{\mathbb{O}}$ is generated by the 2×2 minors.*

Proof. Note that $\overline{\mathbb{O}} = \mathbb{O} \cup \{0\}$. Consider the map:

$$\mu_1 : V \longrightarrow \mathfrak{g}^* \simeq \mathfrak{g}$$

$$v \mapsto v^2.$$

The image of μ_1 is exactly $\overline{\mathbb{O}}$. Also, the pre-image of any point in $\mathbb{O}$ consists of exactly two vectors in V that are negatives of each other, whereas the pre-image of zero is the zero vector. Therefore, we get an induced map from the categorical quotient:

$$\overline{\mu_1} : V//\{\pm 1\} \longrightarrow \overline{\mathbb{O}}.$$

By the above discussion, $\overline{\mu_1}$ is a closed embedding that is a bijection on $\mathbb{C}$-points. Therefore, as the scheme $\overline{\mathbb{O}}$ is reduced, the map $\overline{\mu_1}$ is an isomorphism.

Hence, the coordinate ring $\mathbb{C}[\overline{\mathbb{O}}]$ of $\overline{\mathbb{O}}$ is isomorphic to the invariant ring $\mathbb{C}[V]^{\{\pm 1\}}$. Choosing coordinates $p_1, p_2, \ldots, p_{2n}$ in V, this invariant ring is equal to $\mathbb{C}[p_1, p_2, \ldots, p_{2n}]^{\{\pm 1\}}$. It is clear that this ring is generated by the polynomials $q_{ij} = p_i p_j$ for $1 \leq i, j \leq 2n$. Also, by the second fundamental theorem ([Wey97, Theorem 2.17A]) for the group $O(1) = \{\pm 1\}$, the relations between these generators are exactly given by $R_{ijkl} = q_{ij}q_{kl} - q_{il}q_{kj}$ for $1 \leq i, j, k, l \leq 2n$. Since the pullbacks of these relations R_{ijkl}'s to the coordinate ring of $\mathfrak{g}$ are exactly given by the 2×2 minors, this shows that the defining ideal is exactly generated by these elements. $\square$

Corollary 4.1.2. *The map θ_1 induces an isomorphism of algebras:*

$$\theta_1 : \mathbb{C}[\mathfrak{g}]/K \longrightarrow \mathbb{C}[V]^{\{\pm 1\}} = \mathbb{C}[V]_{even}.$$

The ideal K is the defining ideal of $\overline{\mathbb{O}}$ in $\mathbb{C}[\mathfrak{g}]$ and is generated by the 2×2 minors.

4.1.2 The shifting trick and classical Hamiltonian reduction

The adjoint action of the group $G = Sp(V)$ on the Lie algebra $\mathfrak{g}$ gives rise to a Hamiltonian G-action on the symplectic variety $T^*(\mathfrak{g})$. Corresponding to this action, we get a moment map:

$$\mu_0 : T^*(\mathfrak{g}) \longrightarrow \mathfrak{g}^*.$$

If we identify the space $\mathfrak{g}^*$ with $\mathfrak{g}$ and $T^*(\mathfrak{g})$ with $\mathfrak{g} \times \mathfrak{g}$, the map μ_0 is given explicitly by the commutator map on $\mathfrak{g}$. We can dualize μ_0 to get a co-moment map $\theta_0 : \mathrm{Sym}(\mathfrak{g}) \to \mathbb{C}[T^*(\mathfrak{g})]$.

Also, we have a diagonal G-action on the space $T^*(\mathfrak{g}) \times V$. The moment map $\mu_2 : T^*(\mathfrak{g}) \times V \to \mathfrak{g}^*$ for this action is equal to $\mu_0 + \mu_1$, whereas the co-moment map $\theta_2 : \mathrm{Sym}(\mathfrak{g}) \to \mathbb{C}[T^*(\mathfrak{g}) \times V] \simeq \mathbb{C}[T^*(\mathfrak{g})] \otimes \mathbb{C}[V]$ is defined via $\theta_2(x) = \theta_0(x) \otimes 1 + 1 \otimes \theta_1(x)$ for all $x \in \mathfrak{g}$. Explicitly, the map μ_2 is defined via the formula $(x, y, i) \mapsto [x, y] + i^2$ (cf. Remark 3.1.4).

Now, we can define the schemes that we'll be dealing with:

- We define the scheme $X \subseteq T^*(\mathfrak{g}) \times V$ as the zero fiber of the moment map μ_2. More precisely, the defining ideal I of X in $\mathbb{C}[T^*(\mathfrak{g}) \times V]$ is the one generated by $\theta_2(\mathfrak{g})$ in $\mathbb{C}[T^*(\mathfrak{g}) \times V]$. Using the formula for θ_2 above, this ideal is equal to the one generated by the matrix entries of the expression $[x, y] + i^2$.

- We define the scheme $A \subseteq T^*(\mathfrak{g})$ as the pre-image of $\overline{\mathbb{O}}$ under the moment map μ_0. That is, the defining ideal J of A in $\mathbb{C}[T^*(\mathfrak{g})]$ is the one generated by $\theta_0(K)$. By

Corollary 4.1.2, this ideal is generated by all 2×2 minors of the commutator $[x, y]$ for $(x, y) \in \mathfrak{g} \times \mathfrak{g}$.

There is a projection morphism $\phi : X \to A$ that maps a triple $(x, y, i) \in X$ to the pair $(x, y) \in A$. Let $\{\pm 1\}$ be the center of the group G. We will show that:

Theorem 4.1.3. *The induced morphism on the categorical quotients:*

$$\phi : X//\{\pm 1\} \longrightarrow A//\{\pm 1\} = A,$$

is an isomorphism.

The proof of this theorem will be a consequence of the following linear algebraic lemma, which will also be useful in proving Hamiltonian reduction statements in §4.2.3 and in Chapter 5. The algebras in the following lemma aren't necessarily assumed to be commutative.

Lemma 4.1.4. *Let $\mathcal{A}$ be an associative $\mathbb{C}$-algebra generated by a vector space $\mathcal{V} \subseteq \mathcal{A}$. Suppose the linear map $S : \mathcal{V} \to \mathcal{V}$ that sends $v \mapsto -v$ for $v \in \mathcal{V}$ extends to an algebra anti-automorphism $S : \mathcal{A} \to \mathcal{A}$. Let $\mathcal{B}_0, \mathcal{B}_1$ be $\mathbb{C}$-algebras and fix algebra homomorphisms $f_i : \mathcal{A} \to \mathcal{B}_i$ for $i = 0, 1$. Suppose f_1 is surjective and let $\mathcal{I} = \ker(f_1)$. Let $f_2 : \mathcal{V} \to \mathcal{B}_0 \otimes \mathcal{B}_1$ be the linear map defined via the formula $v \mapsto f_0(v) \otimes 1 + 1 \otimes f_1(v)$. Then, there exists a vector space isomorphism:*

$$\mathcal{B}_0/(\mathcal{B}_0 \cdot f_0(S(\mathcal{I}))) \simeq (\mathcal{B}_0 \otimes \mathcal{B}_1)/((\mathcal{B}_0 \otimes \mathcal{B}_1) \cdot f_2(\mathcal{V})),$$

induced by the linear map $\mathcal{B}_0 \to \mathcal{B}_0 \otimes \mathcal{B}_1$ that sends $b \mapsto b \otimes 1$ for $b \in \mathcal{B}_0$.

Proof. We start by observing that:

$$\mathcal{B}_0/(\mathcal{B}_0 \cdot f_0(S(\mathcal{I}))) \simeq \mathcal{B}_0 \otimes_{\mathcal{A}} (\mathcal{A}/S(\mathcal{I})).$$

By definition, the space $\mathcal{B}_0 \otimes_{\mathcal{A}} (\mathcal{A}/S(\mathcal{I}))$ is equal to the quotient of the algebra $\mathcal{B}_0^{op} \otimes (\mathcal{A}/S(\mathcal{I}))$ by the left ideal generated by elements of the form $f_0(v) \otimes 1 - 1 \otimes v$ for $v \in \mathcal{V}$. Here, we have identified the underlying vector space of the opposite algebra $\mathcal{B}_0^{op}$ with that of the algebra $\mathcal{B}_0$.

Next, we note that the map S induces an algebra anti-isomorphism:

$$S : \mathcal{A}/S(\mathcal{I}) \longrightarrow \mathcal{A}/S(S(\mathcal{I})) = \mathcal{A}/\mathcal{I} \xrightarrow[\sim]{f_1} \mathcal{B}_1 \ .$$

Hence, we can define a map:

$$\mathcal{B}_0 \otimes_{\mathcal{A}} (\mathcal{A}/S(\mathcal{I})) \longrightarrow (\mathcal{B}_0 \otimes \mathcal{B}_1)/((\mathcal{B}_0 \otimes \mathcal{B}_1) \cdot f_2(\mathcal{V})),$$

via the formula $b_0 \otimes a \mapsto b_0 \otimes S(a)$. The fact that S is an anti-isomorphism ensures that the map is well-defined and gives an isomorphism between the two spaces, thus completing the proof. □

Remark 4.1.5. *One can weaken the hypothesis of the lemma and assume that $\mathcal{B}_0$ and $\mathcal{B}_1$ are $\mathcal{A}$-modules (rather than algebras) and prove a slightly modified statement, but we don't need that generality here.*

Proof of Theorem 4.1.3. To prove the isomorphsim of schemes, we need to prove that their respective coordinate rings are isomorphic. That is, we need to show that there is an isomorphism:

$$\phi^* : \mathbb{C}[T^*(\mathfrak{g})]/J = \mathbb{C}[A] \longrightarrow \mathbb{C}[X]^{\{\pm 1\}} = \Big(\mathbb{C}[T^*(\mathfrak{g})] \otimes \mathbb{C}[V])/I\Big)^{\{\pm 1\}},$$

induced by the map $p \mapsto p \otimes 1$ for $p \in \mathbb{C}[T^*(\mathfrak{g})]$. As the group $\{\pm 1\}$ acts trivially on

$\mathbb{C}[T^*(\mathfrak{g})]$, we have the equality of algebras:

$$\left(\mathbb{C}[T^*(\mathfrak{g})] \otimes \mathbb{C}[V])/I\right)^{\{\pm 1\}} = (\mathbb{C}[T^*(\mathfrak{g})] \otimes \mathbb{C}[V]_{even})/(I^{\{\pm 1\}}).$$

We apply Lemma 4.1.4 by taking $\mathcal{A} = \mathrm{Sym}(\mathfrak{g})$, $\mathcal{V} = \mathfrak{g}$, $\mathcal{B}_0 = \mathbb{C}[T^*(\mathfrak{g})]$, $\mathcal{B}_1 = \mathbb{C}[V]_{even}$ and $f_i = \theta_i$ for $i = 0, 1, 2$. The ideal $\mathcal{I} = \ker(f_1) = \ker(\theta_1)$, in this case, is equal to K, which is S-invariant as it is generated by homogeneous polynomials of degree 2 by Corollary 4.1.2. Then, the conclusion of Lemma 4.1.4 gives us the required isomorphism. $\square$

Remark 4.1.6. *Theorem 4.1.3 is a special case of the following 'shifting trick', which is also an immediate consequence of Lemma 4.1.4:*

Let $\mathfrak{g}$ be a reductive Lie algebra. Let G be the corresponding adjoint group and fix $\mathbb{O}$ to be a G-orbit in $\mathfrak{g}^*$ under the co-adjoint action. Let Y be an affine variety with a Hamiltonian G-action. Then, the Hamiltonian reduction of Y at the negative orbit closure $-\overline{\mathbb{O}}$ is isomorphic to the Hamiltonian reduction of the variety $Y \times \overline{\mathbb{O}}$ at $0 \in \mathfrak{g}^*$.

As a consequence of Theorem 4.1.3, we have an isomorphism of schemes $X//G \simeq A//G$. Therefore, we have morphisms:

$$C//G \xrightarrow{\psi} A//G \xleftarrow{\phi} X//G.$$

Set-theoretically, these morphisms are as follows: The map ψ sends a pair (x, y) of commuting matrices to the almost commuting pair (x, y). The map ϕ sends a triple (x, y, i) to the pair (x, y). We have shown that ϕ is an isomorphism. The morphism ψ is an isomorphism too, because it is proven in [Los21] that we have an isomorphism $C//G \to X//G$, and that morphism composed with ϕ gives ψ.

Remark 4.1.7. *The fact that ψ is an isomorphism can also be proven independently by mimicking the proof of Theorem 12.1 of [EG02], making use of Weyl's fundamental theorem of invariant theory for $\mathfrak{g} = \mathfrak{sp}(V)$.*

Combining Theorem 1.1.3 with Theorem 1.3 of [Los21], we have the following corollary:

Corollary 4.1.8. *We have an algebra isomorphism:*

$$\mathbb{C}[\mathfrak{h} \times \mathfrak{h}]^W = \mathbb{C}[(\mathfrak{h} \times \mathfrak{h})//W] \xrightarrow{\sim} \mathbb{C}[A//G] = \left(\mathbb{C}[\mathfrak{g} \times \mathfrak{g}]/J\right)^{\mathfrak{g}}.$$

4.2 Quantum setting

In this section, we prove Theorem 1.1.4. The next section defines the relevant objects, whereas the following two sections provide the proof of the theorem.

4.2.1 Definitions and statement of the main theorem

We set up some notation. Let ω denote the symplectic form on the vector space V. Let L be a fixed Lagrangian subspace of V. Then, we define the Weyl algebra W_{2n} as the algebra of polynomial differential operators on L, also denoted by $\mathcal{D}(L)$. Explicitly, the Weyl algebra is an associative $\mathbb{C}$-algebra generated by the variables $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ that satisfy the following relations:

$$[x_i, x_j] = 0, [y_i, y_j] = 0, [y_i, x_j] = \delta_{i,j}, 1 \leq i, j \leq n.$$

Here, the x_i's correspond to a choice of coordinates on the vector space L^* and and the y_i's represent the partial derivatives ∂_{x_i}'s with respect to x_i's. We have a direct sum decomposition $W_{2n} = W_{2n,even} \oplus W_{2n,odd}$ as vector spaces, where $W_{2n,even}$ is the space spanned by all the monomials in the x_i's and y_i's having even total degree and $W_{2n,odd}$ is the space spanned by the monomials having odd total degree. It is clear that $W_{2n,even}$ is a subalgebra of W_{2n} which is generated by all monomials of the form $x_i x_j, y_i y_j$ and $x_i y_j + y_j x_i$ for $1 \leq i, j \leq n$.

Recall from §4.1 that corresponding to the G action on V, we have a co-moment map:

$$\theta_1 : \mathfrak{g} \longrightarrow \mathbb{C}[V],$$

such that the image of θ_1 in $\mathbb{C}[V]$ is exactly the vector space of polynomial functions on V having degree 2.

Next, we note that the algebra W_{2n} is a quantization of $\mathbb{C}[V]$. To make this statement precise, we define the symmetrization map, which is the following map of vector spaces:

$$\mathrm{Sym} : \mathbb{C}[V] \longrightarrow W_{2n}$$

$$\lambda_1\lambda_2\cdots\lambda_k \mapsto \frac{1}{k!}\sum_{\sigma\in S_k}\lambda_{\sigma(1)}\lambda_{\sigma(2)}\cdots\lambda_{\sigma(k)},$$

where each λ_i is a linear function on V. (Here, we have used that there is a canonical isomorphism of vector spaces $V^* \simeq L \oplus L^*$ using the symplectic form). The map Sym is a vector space isomorphism. Note that both of these spaces have Lie algebra structures, where the Lie bracket on $\mathbb{C}[V]$ is given by the Poisson bracket and the bracket on W_{2n} is the one induced by the commutator of the associative product. The following lemma follows by a straightforward computation:

Lemma 4.2.1. *The restriction of the map* Sym *to the subspace* $\mathbb{C}[V]_{\leq 2} \subseteq \mathbb{C}[V]$ *of polynomials of degree lesser than or equal to two is a Lie algebra homomorphism.*

Now, we define the composition:

$$\Theta_1 := \mathrm{Sym} \circ \theta_1 : \mathfrak{g} \longrightarrow W_{2n}.$$

Then, Θ_1 is a Lie algebra homomorphism, and so it induces an algebra homomorphism $\mathcal{U}\mathfrak{g} \to W_{2n}$, which we also denote by Θ_1. Note that the image of Θ_1 lies in the even subalgebra $W_{2n,even}$. Define $\mathcal{K} := \ker(\Theta_1) \subseteq \mathcal{U}\mathfrak{g}$.

Viewing the Lie algebra $\mathfrak{sp}(V) = \mathfrak{sp}_{2n}$ as a subspace of $\mathfrak{gl}_{2n}$, using the embedding defined in §2.3 the map Θ_1 can be written in terms of coordinates via the formula:

$$\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \mapsto \frac{1}{2}\left(\sum_{i,j=1}^{n} (2a_{i,j}x_iy_j + b_{i,j}x_ix_j - c_{i,j}y_iy_j) + \mathrm{Tr}(A) \right),$$

where $A = (a_{i,j}), B = (b_{i,j})$ and $C = (c_{i,j})$.

Next, we define the action of $\mathfrak{g}$ on the space $\mathcal{D}(\mathfrak{g})$ of polynomial differential operators on $\mathfrak{g}$. For this, fix $x \in \mathfrak{g}$ and consider the linear map:

$$\mathrm{ad}_x : \mathfrak{g} \longrightarrow \mathfrak{g}$$

$$y \mapsto [x, y].$$

Using the identification $\mathfrak{g} \simeq \mathfrak{g}^*$, the assignment $x \mapsto \mathrm{ad}_x$ gives a Lie algebra homomorphism $\mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*)$, where $\mathrm{End}(\mathfrak{g}^*)$ denotes the space of vector space endomorphisms of $\mathfrak{g}^*$. Any element in $\mathrm{End}(\mathfrak{g}^*)$ can be uniquely extended to a derivation on $\mathrm{Sym}(\mathfrak{g}^*) \simeq \mathbb{C}[\mathfrak{g}]$ via Leibniz rule. Since $\mathrm{Der}(\mathbb{C}[\mathfrak{g}]) \subseteq \mathcal{D}(\mathfrak{g})$, we have constructed a map:

$$\Theta_0 : \mathfrak{g} \longrightarrow \mathcal{D}(\mathfrak{g}),$$

which is a Lie algebra homomorphism. Hence, this induces an algebra homomorphism $\mathcal{U}\mathfrak{g} \to \mathcal{D}(\mathfrak{g})$, which we also denote by Θ_0. Then, given $x \in \mathfrak{g}$ and $d \in \mathcal{D}(\mathfrak{g})$, the action of $\mathfrak{g}$ on $\mathcal{D}(\mathfrak{g})$ is defined via:

$$x \cdot d := [\Theta_0(x), d].$$

We now define the algebras we will be working with:

- Consider the algebra $\mathcal{D}(\mathfrak{g}) \otimes W_{2n}$. We have the diagonal $\mathfrak{g}$-action on this algebra:

Given $d \otimes w \in \mathcal{D}(\mathfrak{g}) \otimes W_{2n}$ and $x \in \mathfrak{g}$, we define the action via:

$$x \cdot (d \otimes w) := [\Theta_0(x), d] \otimes w + d \otimes [\Theta_1(x), w].$$

Let $\Theta_2 : \mathfrak{g} \to \mathcal{D}(\mathfrak{g}) \otimes W_{2n}$ be the Lie algebra homomorphism defined via $\Theta_2 = \Theta_0 \otimes 1 + 1 \otimes \Theta_1$. This can be extended to get an algebra homomorphism $\Theta_2 : \mathcal{U}\mathfrak{g} \to \mathcal{D}(\mathfrak{g}) \otimes W_{2n}$, which is the co-moment map for the above action. Then, we have the quantum Hamiltonian reduction $\Big((\mathcal{D}(\mathfrak{g}) \otimes W_{2n})/(\mathcal{D}(\mathfrak{g}) \otimes W_{2n}) \cdot \Theta_2(\mathfrak{g})\Big)^{\mathfrak{g}}$ of $\mathcal{D}(\mathfrak{g}) \otimes W_{2n}$ at the augmentation ideal $\mathcal{U}\mathfrak{g}^+$ of $\mathcal{U}\mathfrak{g}$.

- Define $\mathcal{J} \subseteq \mathcal{D}(\mathfrak{g})$ to be the left ideal generated by the image of $\mathcal{K} \subseteq \mathcal{U}\mathfrak{g}$ under the co-moment map Θ_0, that is, $\mathcal{J} := \mathcal{D}(\mathfrak{g}) \cdot (\Theta_0(\ker(\Theta_1)))$. Then, we have the algebra $\Big(\mathcal{D}(\mathfrak{g})/\mathcal{J}\Big)^{\mathfrak{g}}$ which is the quantum Hamiltonian reduction of the algebra $\mathcal{D}(\mathfrak{g})$ at the ideal $\mathcal{K} \subseteq \mathcal{U}\mathfrak{g}$.

- Finally, we consider the spherical subalgebra eH_ce of the rational Cherednik algebra H_c with the parameter c given by $c = (c_{long}, c_{short}) = (-1/4, -1/2)$.

Our goal is to prove Theorem 1.1.4 by constructing algebra isomorphisms:

$$\Psi : \Big(\mathcal{D}(\mathfrak{g})/\mathcal{J}\Big)^{\mathfrak{g}} \longrightarrow eH_ce,$$
$$\Phi : \Big(\mathcal{D}(\mathfrak{g})/\mathcal{J}\Big)^{\mathfrak{g}} \longrightarrow \Big((\mathcal{D}(\mathfrak{g}) \otimes W_{2n})/(\mathcal{D}(\mathfrak{g}) \otimes W_{2n}) \cdot \Theta_2(\mathfrak{g})\Big)^{\mathfrak{g}}.$$

These isomorphisms are established in Theorems 4.2.7 and 4.2.9 respectively.

Before constructing the maps, we define some filtrations on the algebras we will be working with. On the algebra $\mathcal{D}(\mathfrak{g})$, we have an increasing filtration given by the order of the differential operators. The associated graded with respect to this filtration is given by $\mathrm{gr}(\mathcal{D}(\mathfrak{g})) \simeq \mathbb{C}[T^*(\mathfrak{g})] \simeq \mathbb{C}[\mathfrak{g} \times \mathfrak{g}]$, identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ using the trace pairing. Similarly, we have a filtration on the algebra $\mathcal{D}(\mathfrak{h})$ by the order of differential operators and $\mathrm{gr}(\mathcal{D}(\mathfrak{h})) \simeq$

$\mathbb{C}[\mathfrak{h} \times \mathfrak{h}]$.

On the even part of the Weyl algebra $W_{2n,even}$, we define a version of the Bernstein filtration. Note that the algebra $W_{2n,even}$ is generated by elements of the form $x_i x_j, y_i y_j$ and $x_i y_j + y_j x_i$. We define the filtration on $W_{2n,even}$ by placing each of these elements in degree 1. Thus, any element which is a product of $2m$ of the x_i's and y_i's lies in the m^{th} filtered piece. Under this filtration, we have the associated graded $\mathrm{gr}(W_{2n,even}) = \mathbb{C}[V]_{even}$.

Using the two filtrations above, we also get a tensor product filtration on $\mathcal{D}(\mathfrak{g}) \otimes W_{2n,even}$ such that the associated graded is $\mathrm{gr}(\mathcal{D}(\mathfrak{g}) \otimes W_{2n,even}) = \mathbb{C}[\mathfrak{g} \times \mathfrak{g}] \otimes \mathbb{C}[V]_{even}$. Next, on the algebra $\mathcal{U}\mathfrak{g}$, we have the PBW filtration, and so by PBW theorem, we have the associated graded with respect to this filtration $\mathrm{gr}(\mathcal{U}\mathfrak{g}) \simeq \mathrm{Sym}\,\mathfrak{g}$.

Remark 4.2.2. *Note that the $\mathfrak{g}$-action on each of these algebras is filtration preserving, and so, as $\mathfrak{g}$ is reductive, taking $\mathfrak{g}$-invariants commutes with taking associated graded.*

Finally, the Cherednik algebra H_c has a filtration such that all the elements of W and the generators of $\mathrm{Sym}(\mathfrak{h}^*)$ have degree zero, and the generators of $\mathrm{Sym}(\mathfrak{h})$ have degree one. This also induces a filtration on the spherical subalgebra eH_ce. By Proposition 2.5.2, we have the associated graded $\mathrm{gr}(eH_ce) = \mathbb{C}[\mathfrak{h} \times \mathfrak{h}]^W$.

We prove a lemma that relates the ideals in the non-commutative setting with the ideals in the commutative setting:

Lemma 4.2.3. *We have an inclusion of ideals $J \subseteq \mathrm{gr}(\mathcal{J})$, where $J \subseteq \mathbb{C}[\mathfrak{g} \times \mathfrak{g}]$ is the defining ideal of pairs of matrices whose commutator has rank lesser than or equal to 1.*

Recall from the definitions given in §4.1.2 that the ideal J is generated by all 2×2 minors of the commutator $[x, y]$ for $(x, y) \in \mathfrak{g} \times \mathfrak{g}$.

Proof. First, we describe the associated graded versions of the maps Θ_1 and Θ_0, in order to compute $\mathrm{gr}(\mathcal{J})$. The map $\Theta_1 : \mathcal{U}\mathfrak{g} \to W_{2n,even} \subseteq W_{2n}$ was defined on $\mathfrak{g} \subseteq \mathcal{U}\mathfrak{g}$ as the composition $\mathrm{Sym} \circ \theta_1$. Then, this is a filtration-preserving map, and we have:

$$\theta_1 = \mathrm{gr}(\Theta_1) : \mathrm{Sym}\,\mathfrak{g} = \mathrm{gr}(\mathcal{U}\mathfrak{g}) \longrightarrow \mathrm{gr}(W_{2n,even}) = \mathbb{C}[V]_{even} \subseteq \mathbb{C}[V].$$

This induces a map $V = \mathrm{Spec}(\mathbb{C}[V]) \to \mathrm{Spec}(\mathrm{Sym}\,\mathfrak{g})) = \mathfrak{g}^* \simeq \mathfrak{g}$ that sends a vector v to the rank one endomorphism $v^2 \in \mathfrak{g}$. By Lemma 4.1.1, we know that the ideal $\ker(\theta_1)$ is generated by all 2×2 minors in the entries of $\mathfrak{g}$.

Next, we consider the map $\Theta_0 : \mathcal{U}\mathfrak{g} \to \mathcal{D}(\mathfrak{g})$. In this case, we have the associated graded map:

$$\theta_0 = \mathrm{gr}(\Theta_0) : \mathrm{Sym}\,\mathfrak{g} = \mathrm{gr}(\mathcal{U}\mathfrak{g}) \longrightarrow \mathrm{gr}(\mathcal{D}(\mathfrak{g})) = \mathbb{C}[\mathfrak{g} \times \mathfrak{g}].$$

This is the co-moment map for the diagonal adjoint action of G on $\mathfrak{g} \times \mathfrak{g}$. This morphism θ_0 induces a map $\mathfrak{g} \times \mathfrak{g} = \mathrm{Spec}(\mathbb{C}[\mathfrak{g} \times \mathfrak{g}]) \to \mathrm{Spec}(\mathrm{Sym}\,\mathfrak{g}) = \mathfrak{g}^* \simeq \mathfrak{g}$ sending a pair (x, y) to the commutator $[x, y]$.

Hence, if we consider the ideal generated by the image of $\ker(\theta_1)$ under the map θ_0, then this is exactly generated by the 2×2 minors of the commutator $[x, y]$ for $(x, y) \in \mathfrak{g} \times \mathfrak{g}$. But these are exactly the generators of the ideal J, and so $J = \mathbb{C}[\mathfrak{g} \times \mathfrak{g}] \cdot \theta_0(\ker(\theta_1))$. Also, by definition, $\mathcal{J} = \mathcal{D}(\mathfrak{g}) \cdot \Theta_0(\ker(\Theta_1))$. Hence, to prove that $J \subseteq \mathrm{gr}(\mathcal{J})$, it suffices to show that $\mathrm{gr}(\ker(\Theta_1)) = \ker(\theta_1)$. It is clear that $\mathrm{gr}(\ker(\Theta_1)) \subseteq \ker(\theta_1)$.

To see that the inclusion is an equality, we first describe the images of the maps θ_1 and Θ_1. We have that $\theta_1(\mathfrak{g})$ is exactly the space of degree 2 polynomials in $\mathbb{C}[V]$, and so, $\mathrm{Im}(\theta_1)$ is equal to the subalgebra $\mathbb{C}[V]_{even}$ of all polynomials having even total degree. Applying the symmetrization map, we see that $\mathrm{Im}(\Theta_1)$ is exactly $W_{2n,even}$. Thus, we have a short exact sequence of filtered $\mathcal{U}\mathfrak{g}$-modules:

$$0 \longrightarrow \ker(\Theta_1) \longrightarrow \mathcal{U}\mathfrak{g} \longrightarrow W_{2n,even} \longrightarrow 0.$$

We claim that the filtrations on $\ker(\Theta_1)$ and $W_{2n,even}$ are exactly the ones induced on them by $\mathcal{U}\mathfrak{g}$. For $\ker(\Theta_1)$, this is true by definition. For $W_{2n,even}$, this follows by observing that the function Θ_1 maps $\mathfrak{g} \subseteq \mathcal{U}\mathfrak{g}$ to exactly the space spanned by the elements $x_i x_j, y_i y_j$ and $x_i y_j + y_j x_i$. Hence, this is a strict exact sequence of filtered modules, and so, by [Sjö73, Lemma 1], we can take the associated graded to get an exact sequence of $\mathrm{gr}(\mathcal{U}\mathfrak{g}) = \mathrm{Sym}\,\mathfrak{g}$-modules:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{gr}(\ker(\Theta_1)) \arrow[r] \arrow[d, hook] & \mathrm{gr}(\mathcal{U}\mathfrak{g}) \arrow[r] \arrow[d, equal] & \mathrm{gr}(W_{2n,even}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker(\theta_1) \arrow[r] & \mathrm{Sym}\,\mathfrak{g} \arrow[r] & \mathbb{C}[V]_{even} \arrow[r] & 0
\end{tikzcd}.$$

This implies that $\mathrm{gr}(\ker(\Theta_1)) = \ker(\theta_1)$, completing the proof. □

Remark 4.2.4. *The ideal $\mathcal{K} = \ker(\Theta_1)$ is a primitive ideal of $\mathcal{U}\mathfrak{g}$ and the equality of ideals $\mathrm{gr}(\ker(\Theta_1)) = \ker(\theta_1)$ can also be seen in a more general setting in the works of Joseph, where he constructs minimal realizations of simple Lie algebras in the Weyl algebra (see [Jos74], [Jos76]). This ideal is often referred to as the Joseph ideal in the literature.*

4.2.2 Construction of the isomorphism Ψ

In this section, we construct the isomorphism:

$$\Psi : \Big(\mathcal{D}(\mathfrak{g})/\mathcal{J}\Big)^{\mathfrak{g}} \longrightarrow eH_c e.$$

For this, we recall the map $\Theta_1 : \mathfrak{g} \to W_{2n}$ defined in the previous section. The Weyl algebra W_{2n} is the space of polynomial differential operators on the vector space L, and we have fixed coordinates $x_1, x_2, \ldots, x_n$ on L^*. Let $\mathfrak{V}$ be the vector space spanned by all expressions of the form $(x_1 x_2 \cdots x_n)^{-1/2} \cdot P$, where P is a Laurent polynomial in $x_1, x_2, \ldots, x_n$, i.e. $P \in \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \cdots, x_n^{\pm 1}]$. Then, the map Θ_1 gives an action of $\mathfrak{g}$ on $\mathfrak{V}$, where any element $f \in \mathfrak{g}$ acts on $\mathfrak{V}$ by $\Theta_1(f) \in W_{2n}$ via formal differentiation of Laurent polynomials.

Recall that the Cartan $\mathfrak{h} \subseteq \mathfrak{g}$ is spanned by elements of the form $E_{i,i} - E_{n+i,n+i}$ for $1 \leq i \leq n$. Then, under Θ_1, the image of this element is the differential operator $(x_i y_i + y_i x_i)/2 = x_i y_i + \frac{1}{2} = x_i \partial_{x_i} + \frac{1}{2}$. Hence, the $\mathfrak{g}$-action on $\mathfrak{V}$ has a one-dimensional zero weight space $\mathfrak{V}\langle 0 \rangle$ spanned by the element $(x_1 x_2 \cdots x_n)^{-1/2} \cdot 1$. Then, using the construction of the radial parts homomorphism in §2.4, we get an algebra homomorphism:

$$\Psi : \mathcal{D}(\mathfrak{g})^{\mathfrak{g}} \longrightarrow \mathcal{D}(\mathfrak{h}^{reg})^W.$$

Let $\Delta_{\mathfrak{g}}$ and $\Delta_{\mathfrak{h}}$ be the Laplacian operators on $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Recall from §2.5 the Caloger-Moser differential operator L_c for $c = (c_{long}, c_{short}) = (-1/4, -1/2)$ and let $\mathcal{C}_c$ denote the centralizer of L_c in $\mathcal{D}(\mathfrak{h}^{reg})^W$. Also, let $\mathrm{Sym}(\mathfrak{g}) \subseteq \mathcal{D}(\mathfrak{g})$ denote the subalgebra of differential operators on $\mathfrak{g}$ with constant coefficients.

The computations in the following lemma are the origin of the precise value of our parameter c:

Lemma 4.2.5. *1. We have the equality:*

$$\Psi(\Delta_{\mathfrak{g}}) = L_c.$$

2. The map Ψ induces an isomorphism:

$$\mathrm{Sym}(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} C_c.$$

Proof. 1. By Proposition 6.2 of [EG02], we have:

$$\Psi(\Delta_{\mathfrak{g}}) = \Delta_{\mathfrak{h}} - \sum_{\alpha \in R} \frac{e_\alpha \cdot e_{-\alpha}}{\alpha^2}.$$

We will evaluate $e_\alpha \cdot e_{-\alpha}|_{\mathfrak{V}\langle 0 \rangle}$ for the long roots and the short roots separately:

(a) Suppose α is a long root. (For such roots, $(\alpha, \alpha) = 2$.) Hence, $\alpha = \sqrt{2} r_i$ for some i, where $r_i = (E_{i,i} - E_{i+n,i+n})/\sqrt{2}$. Then, $e_\alpha = e_{-\alpha}^T = E_{i,i+n}$, which implies that:

$$\Theta_1(e_\alpha \cdot e_{-\alpha}) = \frac{(x_i{}^2) \cdot (-\partial_{x_i}^2)}{4}.$$

It is straightforward to see that this operator acts by $\frac{-3}{16} \operatorname{Id}_{\mathfrak{V}\langle 0\rangle} = c_{long}(c_{long} + 1) \operatorname{Id}_{\mathfrak{V}\langle 0\rangle}$ on the space spanned by $(x_1 x_2 \cdots x_n)^{-1/2} \cdot 1$.

(b) Suppose α is a short root. (For such roots, $(\alpha, \alpha) = 1$.) Then, $\alpha = (r_i + r_j)/\sqrt{2}$ or $\alpha = (r_i - r_j)/\sqrt{2}$ for some $i \neq j$. In the first case, $e_\alpha = e_{-\alpha}^T = (E_{i,j+n} + E_{j,i+n})/\sqrt{2}$, and so:

$$\Theta_1(e_\alpha \cdot e_{-\alpha}) = \frac{(x_i x_j) \cdot (-\partial_{x_i} \partial_{x_j})}{2}.$$

In the second case, $e_\alpha = e_{-\alpha}^T = (E_{i,j} + E_{j+n,i+n})/\sqrt{2}$, giving that:

$$\Theta_1(e_\alpha \cdot e_{-\alpha}) = \frac{(x_i \partial_{x_j}) \cdot (x_j \partial_{x_i})}{2}.$$

Then, in either case, the differential operator acts by $\frac{-1}{8} \operatorname{Id}_{\mathfrak{V}\langle 0\rangle} = \frac{1}{2} c_{short}(c_{short} + 1) \operatorname{Id}_{\mathfrak{V}\langle 0\rangle}$ on the space spanned by $(x_1 x_2 \cdots x_n)^{-1/2} \cdot 1$.

Hence, we conclude that:

$$\Psi(\Delta_{\mathfrak{g}}) = \Delta_{\mathfrak{h}} - \frac{1}{2} \sum_{\alpha \in R} \frac{c(\alpha)(c(\alpha)+1)}{\alpha^2} \cdot (\alpha, \alpha),$$

which is exactly the Calogero-Moser operator L_c of Type C for the parameter c.

2. This follows from the proof of Proposition 7.2 of [EG02], which works for any reductive Lie algebra $\mathfrak{g}$.

□

By the above lemma, we see that $\mathcal{C}_c \subseteq \mathrm{Im}(\Psi)$. Next, the restriction of Ψ to $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \subseteq \mathcal{D}(\mathfrak{g})^{\mathfrak{g}}$ is exactly the Chevalley restriction map $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{h}]^W \subseteq \mathcal{D}(\mathfrak{h}^{reg})$, and so, $\mathbb{C}[\mathfrak{h}]^W \subseteq \mathrm{Im}(\Psi)$. Now, by Theorem 2.5.1, the image of the spherical subalgebra eH_ce under the Dunkl homomorphism $\Theta : eH_ce \to \mathcal{D}(\mathfrak{h}^{reg})^W$ is exactly the subalgebra of $\mathcal{D}(\mathfrak{h}^{reg})^W$ generated by $\mathcal{C}_c$ and $\mathbb{C}[\mathfrak{h}]^W$. Hence, we have the following corollary:

Corollary 4.2.6. *We have the inclusion of algebras* $\Theta(eH_ce) \subseteq \mathrm{Im}(\Psi)$.

Next, let $\mathrm{Ann}(\mathfrak{V}) \subseteq \mathcal{U}\mathfrak{g}$ denote the annihilator of the representation $\mathfrak{V}$. Then, as remarked in §2.4, we have that $(\mathcal{D}(\mathfrak{g}) \cdot \mathrm{ad}(\mathrm{Ann}(\mathfrak{V})))^{\mathfrak{g}} \subseteq \ker(\Psi)$. Finally, we note that the action of $\mathfrak{g}$ on $\mathfrak{V}$ was defined via the map Θ_1, and so, $\ker(\Theta_1) \subseteq \mathrm{Ann}(\mathfrak{V})$, which implies that $\mathcal{J}^{\mathfrak{g}} = (\mathcal{D}(\mathfrak{g}) \cdot \Theta_0(\ker(\Theta_1)))^{\mathfrak{g}} \subseteq (\mathcal{D}(\mathfrak{g}) \cdot \mathrm{ad}(\mathrm{Ann}(\mathfrak{V})))^{\mathfrak{g}}$.

Theorem 4.2.7. *We have an isomorphism of algebras:*

$$\Big(\mathcal{D}(\mathfrak{g})/\mathcal{J}\Big)^{\mathfrak{g}} \simeq eH_ce.$$

Proof. The proof of this theorem is based on a commutative diagram that is quite similar to the one present in the proof of Theorem 1.3.1 of [GG06].

As noted above, we have $\mathcal{J}^{\mathfrak{g}} \subseteq (\mathcal{D}(\mathfrak{g}) \cdot \mathrm{ad}(\mathrm{Ann}(\mathfrak{V})))^{\mathfrak{g}} \subseteq \ker(\Psi)$, and so, the map Ψ can be factored to induce an algebra hommomorphism (which we also denote by Ψ):

$$\Psi : \Big(\mathcal{D}(\mathfrak{g})/\mathcal{J}\Big)^{\mathfrak{g}} \longrightarrow \mathcal{D}(\mathfrak{h}^{reg}).$$

Identifying $\mathfrak{h}$ with $\mathfrak{h}^*$ using the trace form, we get an algebra isomorphism $\phi : \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*] \to$

$\mathbb{C}[\mathfrak{h} \times \mathfrak{h}]$. Then, we have the following diagram:

$$\begin{tikzcd}
\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]^W \arrow[r, "\phi", "\sim"'] \arrow[d, "\text{Proposition 2.5.2}"', "\sim"] & \mathbb{C}[\mathfrak{h} \times \mathfrak{h}]^W \arrow[r, "\text{Cor. 4.1.8}", "\sim"'] & \Big(\mathbb{C}[\mathfrak{g} \times \mathfrak{g}]/J\Big)^{\mathfrak{g}} \arrow[d, two heads, "\text{Lemma 4.2.3}"] \\
\mathrm{gr}(eH_c e) \arrow[d, "\text{Proposition 2.5.2}"', "\sim"] & & \Big(\mathrm{gr}(\mathcal{D}(\mathfrak{g}))/\mathrm{gr}(\mathcal{J})\Big)^{\mathfrak{g}} \arrow[d, two heads, "\text{proj}"] \\
\mathrm{gr}(\Theta(eH_c e)) \arrow[r, hook, "\text{Cor. 4.2.6}"] & \mathrm{gr}\Big(\Psi\Big(\mathcal{D}(\mathfrak{g})/\mathcal{J}\Big)^{\mathfrak{g}}\Big) & \mathrm{gr}\Big(\mathcal{D}(\mathfrak{g})/\mathcal{J}\Big)^{\mathfrak{g}} \arrow[l, two heads, "\mathrm{gr}(\Psi)"']
\end{tikzcd}.$$

(That the rightmost map in the bottom row is a surjection follows from Proposition 6.1 of [EG02].)

This diagram commutes, and so, we get that all the arrows must be bijections. In particular, the image of $\mathrm{gr}(\Psi)$ in $\mathrm{gr}(\mathcal{D}(\mathfrak{h}^{reg})) = \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^{reg}]$ can be identified with the image of $\mathrm{gr}(\Theta)$. This identification can also be obtained as the associated graded of the embedding $\Theta(eH_c e) \subseteq \mathrm{Im}(\Psi)$ from Corollary 4.2.6, and so, this embedding must itself be an equality. Hence, we can compose with Θ^{-1} (as Θ is injective) to get an algebra homomorphism:

$$\Theta^{-1} \circ \Psi : \Big(\mathcal{D}(\mathfrak{g})/\mathcal{J}\Big)^{\mathfrak{g}} \longrightarrow eH_c e.$$

It follows from the commutative diagram that the associated graded version of this map gives the bijection between $\mathrm{gr}\Big((\mathcal{D}(\mathfrak{g})/\mathcal{J})^{\mathfrak{g}}\Big)$ and $\mathrm{gr}(eH_c e)$. Hence, the map $\Theta^{-1} \circ \Psi$ itself must be a bijection, which is exactly the claim of the theorem. $\square$

Corollary 4.2.8. *1. We have an isomorphism of commutative algebras:*

$$\mathrm{gr}\Big(\mathcal{D}(\mathfrak{g})/\mathcal{J}\Big)^{\mathfrak{g}} \xrightarrow{\sim} \mathrm{gr}(eH_c e).$$

2. All the maps in the above commutative diagram are isomorphisms. In particular, we get that:

(a) We have an isomorphism:

$$\mathbb{C}[A//G] = \Big(\mathbb{C}[\mathfrak{g}\times\mathfrak{g}]/J\Big)^{\mathfrak{g}} \simeq \operatorname{gr}\Big(\mathcal{D}(\mathfrak{g})/\mathcal{J}\Big)^{\mathfrak{g}}.$$

(b) We have the equality of ideals $\operatorname{gr}(\mathcal{J})^{\mathfrak{g}} = J^{\mathfrak{g}}$ *in the ring* $\mathbb{C}[\mathfrak{g}\times\mathfrak{g}]^{\mathfrak{g}}$ *(cf. Lemma 4.2.3).*

4.2.3 Construction of the isomorphism Φ

Now, we construct an isomorphism of non-commutative algebras:

$$\Phi : \Big(\mathcal{D}(\mathfrak{g})/\mathcal{J}\Big)^{\mathfrak{g}} \longrightarrow \Big((\mathcal{D}(\mathfrak{g})\otimes W_{2n})/(\mathcal{D}(\mathfrak{g})\otimes W_{2n})\cdot\Theta_2(\mathfrak{g})\Big)^{\mathfrak{g}}.$$

In fact, we prove the following stronger result:

Theorem 4.2.9. *There is an isomorphism of vector spaces:*

$$\Phi : \mathcal{D}(\mathfrak{g})/\mathcal{J} \longrightarrow \Big((\mathcal{D}(\mathfrak{g})\otimes W_{2n})/(\mathcal{D}(\mathfrak{g})\otimes W_{2n})\cdot\Theta_2(\mathfrak{g})\Big)^{\{\pm 1\}}.$$

Furthermore, the map Φ *restricts to an algebra isomorphism between the respective subspaces of* $\mathfrak{g}$*-invariants.*

Proof. As the group $\{\pm 1\}$ acts trivially on the algebra $\mathcal{D}(\mathfrak{g})$, we have an equality of vector spaces:

$$\Big((\mathcal{D}(\mathfrak{g})\otimes W_{2n})/(\mathcal{D}(\mathfrak{g})\otimes W_{2n})\cdot\Theta_2(\mathfrak{g})\Big)^{\{\pm 1\}} = (\mathcal{D}(\mathfrak{g})\otimes W_{2n,even})/(\mathcal{D}(\mathfrak{g})\otimes W_{2n,even})\cdot\Theta_2(\mathfrak{g}).$$

We apply Lemma 4.1.4 taking $\mathcal{A} = \mathcal{U}\mathfrak{g}$, $\mathcal{V} = \mathfrak{g}$, $\mathcal{B}_0 = \mathcal{D}(\mathfrak{g})$, $\mathcal{B}_1 = W_{2n,even}$ and $f_i = \Theta_i$ for $i = 0, 1, 2$. Here, the ideal $\mathcal{I} = \ker(f_1) = \ker(\Theta_1)$ is equal to $\mathcal{K}$, which is S-invariant because we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{U}\mathfrak{g} & \xrightarrow{\Theta_1} & W_{2n} \\ {\scriptstyle S}\big\downarrow & & \big\downarrow{\scriptstyle S'} \\ \mathcal{U}\mathfrak{g} & \xrightarrow{\Theta_1} & W_{2n} \end{array},$$

where the right vertical map $S' : W_{2n} \to W_{2n}$ is an algebra anti-homomorphism defined by sending the generators $x_1, \dots, x_n, y_1, \dots, y_n$ of W_{2n} to $ix_1, \dots ix_n, iy_1, \dots, iy_n$ respectively, where $i = \sqrt{-1}$. Then, the conclusion from Lemma 4.1.4 gives us the required vector space isomorphism.

Furthermore, as this isomorphism is induced by the map $\mathcal{D}(\mathfrak{g}) \to \mathcal{D}(\mathfrak{g}) \otimes W_{2n}$ given by $D \mapsto D \otimes 1$ for $D \in \mathcal{D}(\mathfrak{g})$, which is an algebra homomorphism, the restriction to $\mathfrak{g}$-invariants gives an algebra isomorphism. $\square$

Corollary 4.2.10. *We have isomorphisms of commutative algebras:*

$$\operatorname{gr}\Big(\mathcal{D}(\mathfrak{g})/\mathcal{J}\Big)^{\mathfrak{g}} \simeq \operatorname{gr}\Big((\mathcal{D}(\mathfrak{g}) \otimes W_{2n})/(\mathcal{D}(\mathfrak{g}) \otimes W_{2n}) \cdot \Theta_2(\mathfrak{g})\Big)^{\mathfrak{g}} \simeq \Big(\mathbb{C}[\mathfrak{g} \times \mathfrak{g} \times V]/I\Big)^{\mathfrak{g}} = \mathbb{C}[X//G].$$

Proof. Recall from §4.1 that we have an isomorphism of schemes $\phi : X//G \to A//G$. This gives an isomorphism between the coordinate rings:

$$\phi^* : \Big(\mathbb{C}[\mathfrak{g} \times \mathfrak{g}]/J\Big)^{\mathfrak{g}} = \mathbb{C}[A//G] \longrightarrow \mathbb{C}[X//G] = \Big(\mathbb{C}[\mathfrak{g} \times \mathfrak{g} \times V]/I\Big)^{\mathfrak{g}}.$$

Hence, by Corollary 4.2.8, we have an isomorphism:

$$\operatorname{gr}\Big(\mathcal{D}(\mathfrak{g})/\mathcal{J}\Big)^{\mathfrak{g}} \simeq \mathbb{C}[A//G] \simeq \mathbb{C}[X//G]$$

Then, we can consider the commutative diagram:

$$\begin{tikzcd}
\Big(\mathbb{C}[\mathfrak{g}\times\mathfrak{g}]/J\Big)^{\mathfrak{g}} \arrow[r, "\phi^*", "\sim"'] \arrow[d, two heads, "proj"] & \Big(\mathbb{C}[\mathfrak{g}\times\mathfrak{g}\times V]/I\Big)^{\mathfrak{g}} \arrow[d, two heads, "proj"] \\
\mathrm{gr}\Big(\mathcal{D}(\mathfrak{g})/\mathcal{J}\Big)^{\mathfrak{g}} \arrow[r, "\mathrm{gr}(\Phi)"] & \mathrm{gr}\Big((\mathcal{D}(\mathfrak{g})\otimes W_{2n})/(\mathcal{D}(\mathfrak{g})\otimes W_{2n})\cdot\Theta_2(\mathfrak{g})\Big)^{\mathfrak{g}}
\end{tikzcd},$$

In this diagram, the top and the left maps are already known to be bijective. We have now shown that Φ is bijective too. Also, it's clear by unwrapping the defintions that Φ is a filtration preserving map, and so is its inverse. Therefore, we conclude that $\mathrm{gr}(\Phi)$ must also be a bijection. This forces the fourth map in the commutative diagram to be a bijection. $\square$

4.2.4 Quantum Hamiltonian reduction functor

Let $(\mathcal{D}(\mathfrak{g})\otimes W_{2n})$-*mod* denote the category of finitely generated $(\mathcal{D}(\mathfrak{g})\otimes W_{2n})$-modules. The algebra $\mathcal{D}(\mathfrak{g})\otimes W_{2n}$ contains the subalgebra $Z = \mathrm{Sym}(\mathfrak{g})^{\mathfrak{g}}$ of invariant differential operators on $\mathfrak{g}$ with constant coefficients. Let $Z_+ \subseteq Z$ be the augmentation ideal, consisting of differential operators with zero constant term.

Furthermore, we have the algebra homomorphism $\Theta_2 : \mathcal{U}\mathfrak{g} \to \mathcal{D}(\mathfrak{g})\otimes W_{2n}$. Also, let $eu \in \mathcal{D}(\mathfrak{g}) \subseteq \mathcal{D}(\mathfrak{g})\otimes W_{2n}$ denote the Euler vector field on $\mathfrak{g}$. Let $\mathcal{U}$ be the subalgebra of $\mathcal{D}(\mathfrak{g})\otimes W_{2n}$ generated by the image of Θ_2 and eu.

Definition 4.2.1. Let $\mathcal{C}$ be the full subcategory of $(\mathcal{D}(\mathfrak{g})\otimes W_{2n})$-*mod*, whose objects are $(\mathcal{D}(\mathfrak{g})\otimes W_{2n})$-modules M, such that the action on M of the subalgebra Z_+ is locally nilpotent and the action of $\mathcal{U}$ is locally finite. The category $\mathcal{C}$ will be referred to as the category of admissible $(\mathcal{D}(\mathfrak{g})\otimes W_{2n})$-modules.

For any $M \in \mathcal{C}$, we have a $\mathfrak{g}$-action on M via the map Θ_2. As the group $G = Sp(V)$ is simply connected, the action of $\mathcal{U}$ on M being locally finite implies that the $\mathfrak{g}$-action on M can be integrated to get a rational representation of the group G on M. Thus, the local finiteness condition in the above definition implies that the modules M are G-equivariant.

We can identify $\mathcal{D}(\mathfrak{g}) \otimes W_{2n}$ with the ring of differential operators $\mathcal{D}(\mathfrak{g} \times L)$. Taking the order filtration on this algebra, for any finitely generated $\mathcal{D}(\mathfrak{g}) \otimes W_{2n}$-module M, there exists a characteristic variety $\mathrm{Ch}(M) \subseteq T^*(\mathfrak{g} \times L) \simeq \mathfrak{g} \times \mathfrak{g} \times V$.

Proposition 4.2.11. *For any G-equivariant module $M \in (\mathcal{D}(\mathfrak{g}) \otimes W_{2n})$-mod, we have $M \in \mathcal{C}$ if and only if $\mathrm{Ch}(M) \subseteq X^{nil}$. Furthermore, all the objects in $\mathcal{C}$ are holonomic $(\mathcal{D}(\mathfrak{g}) \otimes W_{2n})$-modules.*

Proof. The proof of the fact that $M \in \mathcal{C}$ if and only if $\mathrm{Ch}(M) \subseteq X^{nil}$ follows by essentially repeating the proof of Proposition 5.3.2 of [GG06] replacing $\mathfrak{gl}(V) \times \mathbb{P}$ by $\mathfrak{sp}(V) \times L$ everywhere. The holonomicity of the objects in $\mathcal{C}$ follows from the fact that their characteristic variety lies in X^{nil}, which is a Lagrangian subvariety of $\mathfrak{g} \times \mathfrak{g} \times V$. □

We now define the quantum Hamiltonian reduction functor. Let Q be the quotient $(\mathcal{D}(\mathfrak{g}) \otimes W_{2n})/((\mathcal{D}(\mathfrak{g}) \otimes W_{2n}) \cdot \Theta_2(\mathfrak{g}))$. We have the quantum Hamiltonian reduction of $\mathcal{D}(\mathfrak{g}) \otimes W_{2n}$ with respect to the $\mathfrak{g}$-action, given by $\mathcal{A} := Q^{\mathfrak{g}} = \Big((\mathcal{D}(\mathfrak{g}) \otimes W_{2n})/(\mathcal{D}(\mathfrak{g}) \otimes W_{2n}) \cdot \Theta_2(\mathfrak{g})\Big)^{\mathfrak{g}}$. By Theorem 1.1.4, we have an isomorphism $\mathcal{A} \simeq eH_ce$, where eH_ce is the spherical Cherednik algebra with parameter $c = (-1/4. - 1/2)$. Let eH_ce-*mod* be the category of finitely generated eH_ce-modules. Then, by Proposition 7.2.2 and Corollary 7.2.4 of [GG06], we have:

Proposition 4.2.12. *1. The space Q is a finitely generated $(\mathcal{D}(\mathfrak{g}) \otimes W_{2n})$-module.*

2. There is an exact functor $\mathbb{H}$:

$$\mathbb{H} : (\mathcal{D}(\mathfrak{g}) \otimes W_{2n})\text{-}mod \longrightarrow eH_ce\text{-}mod$$

$$M \mapsto \mathrm{Hom}_{\mathcal{D}(\mathfrak{g}) \otimes W_{2n}}(Q, M) = M^{\mathfrak{g}}.$$

3. *The functor* $\mathbb{H}$ *has a left adjoint* ${}^T\mathbb{H}$*:*

$$ {}^T\mathbb{H} : eH_ce\text{-}mod \longrightarrow (\mathcal{D}(\mathfrak{g}) \otimes W_{2n})\text{-}mod $$

$$ M \mapsto Q \otimes_{\mathcal{A}} M, $$

   such that the canonical adjunction morphism $M \to {}^T\mathbb{H}(\mathbb{H}(M))$ *is an isomorphism for all* $M \in (\mathcal{D}(\mathfrak{g}) \otimes W_{2n})$*-mod.*

4. *The full subcategory* $\ker(\mathbb{H})$ *is a Serre subcategory of* $(\mathcal{D}(\mathfrak{g}) \otimes W_{2n})$*-mod and the functor* $\mathbb{H}$ *induces an equivalence of categories:*

$$ (\mathcal{D}(\mathfrak{g}) \otimes W_{2n})\text{-}mod/\ker(\mathbb{H}) \simeq eH_ce\text{-}mod $$

Next, we recall that the spherical subalgebra eH_ce contains the subalgebra $\mathrm{Sym}(\mathfrak{h})^W$. Let $\mathrm{Sym}(\mathfrak{h})^W_+$ denote the augmentation ideal of $\mathrm{Sym}(\mathfrak{h})^W$. Also recall the category $\mathcal{O}(H_c) = \mathcal{O}(eH_ce)$, which is the full subcategory of eH_ce-*mod* whose objects are finitely generated eH_ce-modules with locally nilpotent action of $\mathrm{Sym}(\mathfrak{h})^W_+ \subseteq eH_ce$.

Proposition 4.2.13. *The functor* $\mathbb{H}$ *restricts to an exact functor* $\mathbb{H} : \mathcal{C} \to \mathcal{O}(eH_ce)$*. This induces an equivelence of categories* $\mathcal{C}/\ker(\mathbb{H}) \simeq \mathcal{O}(eH_ce)$*.*

Proof. Under the isomorphism $\mathcal{A} \simeq eH_ce$, the subalgebra Z_+ of $\mathcal{A}$ is mapped exactly to the subalgebra $\mathrm{Sym}(\mathfrak{h})^W_+$ of eH_ce. Therefore, for any $M \in \mathcal{C}$, we have $\mathbb{H}(M) \in \mathcal{O}(eH_ce)$. Furthermore, by the corollary to Lemma 2.5 of [BEG03a], there exists an element $h \in H_c$ that acts locally finitely on every element of $\mathcal{O}(H_c) \simeq \mathcal{O}(eH_ce)$, such that the image of the Euler vector field eu in eH_ce is equal to h upto an additive scalar factor (see the proof of Formula 6.7 in [BEG03a]). Thus, the local finiteness condition on the eu-action in the definition of $\mathcal{C}$ is automatically true in $\mathcal{O}(eH_ce)$. $\square$

CHAPTER 5

GROUP-THEORETIC ANALOG

In this chapter, we note the group-theoretic analogs of the results proven in the last two chapters. Most of these results can be proved by repeating the proofs in the Lie algebra setting with minor modifications that we will point out.

5.1 Almost commuting scheme and nilpotent subscheme

As before, let V be a symplectic vector space over $\mathbb{C}$ of dimension $2n$ with symplectic form ω and let $G = Sp(V)$. We consider the scheme $T^*(G) \times V$, which is naturally isomorphic to $G \times \mathfrak{g}^* \times V \simeq G \times \mathfrak{g} \times V$, where $\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{sp}(V)$. The group G acts diagonally on $G \times \mathfrak{g} \times V$ and this action is Hamiltonian with moment map given by:

$$\mu : G \times \mathfrak{g} \times V \longrightarrow \mathfrak{g}$$

$$(g, x, i) \mapsto \mathrm{Ad}g(x) - x + i^2.$$

Motivated by the definition in [Los21], we define the group-theoretic analog $\mathfrak{X}$ of the almost commuting scheme to be the zero fiber of this moment map. More precisely, $\mathfrak{X}$ is the (not necessarily reduced) closed subscheme of $T^*(G) \times V$ whose defining equations are given by the matrix entries of the expression $\mathrm{Ad}g(x) - x + i^2$. In this section, we are going to study the properties of this scheme $\mathfrak{X}$ and, in particular, show that it is irreducible and a reduced complete intersection of dimension $\dim(\mathfrak{g}) + \dim(V)$.

We define the scheme $\mathfrak{X}^{nil}$ to be the reduced subscheme of $\mathfrak{X}$ whose $\mathbb{C}$-points are exactly those for which y is nilpotent. That is,

$$\mathfrak{X}^{nil} := \{(g, y, i) \in G \times \mathfrak{g} \times V : \mathrm{Ad}g(x) - x + i^2 \text{ and } y \text{ is nilpotent}\}.$$

Remark 5.1.1. *An analogous scheme was mentioned in the Type A setting in [GG06, Remark 5.3.6].*

Theorem 5.1.2. *The scheme $\mathfrak{X}^{nil}$ is a Lagrangian complete intersection in the symplectic variety $T^*(G) \times V$.*

The proof of this theorem mimics that of Theorem 3.1.1 by embedding $\mathfrak{X}^{nil}$ into the Lagrangian subscheme of $T^*(GL(V) \times V)$ constructed in [FG10a, Lemma 4.4.1]. As a corollary of this theorem, we have:

Corollary 5.1.3. *The scheme $\mathfrak{X}$ is a complete intersection.*

We'll prove another corollary of Theorem 5.1.2 that will allow us to prove the reducedness of $\mathfrak{X}$. For this, we recall a construction discussed in Chapter 3: Fix a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ and let W denote the Weyl group. Then, we can consider the composition $\phi : \mathfrak{g} \to \mathfrak{g}//G \to \mathfrak{h}//W$, where the first map is the quotient and the second one is the Chevalley restriction isomorphism. Then, we can consider the morphism:

$$\pi : \mathfrak{X} \longrightarrow \mathfrak{h}//W,$$

that sends a triple (g, y, i) to $\phi(y)$. It is clear that $\mathfrak{X}^{nil} = \pi^{-1}(\{0\})$.

Proposition 5.1.4. *All fibers of the map π have dimension exactly* $\dim(\mathfrak{g}) + \frac{1}{2}\dim(V)$.

Proof. For any $x \in \mathfrak{h}//W$, we have the dimension inequality:

$$\dim(\pi^{-1}(\{x\})) \geq \dim(\mathfrak{X}) - \dim(\mathfrak{h}//W) = \dim(\mathfrak{g}) + \frac{1}{2}\dim(V).$$

Next, we use an argument based on the asymptotic cone construction as described in the Proof of Proposition 2.3.2 of [GG06]. We have a $\mathbb{C}^*$-action on $\mathfrak{X}$ such that $z \in \mathbb{C}^*$ acts on G and $\mathfrak{g}$ by scaling by z^2 and on V by multiplication of z. This makes π a $\mathbb{C}^*$-equivariant map.

This allows us to conclude (see, for example, [CG10, §2.3.9]) that for any $x \in \mathfrak{h}//W$, we must have $\dim(\pi^{-1}(\{x\})) \leq \dim(\pi^{-1}(\{0\}) = \dim(\mathfrak{g}) + \frac{1}{2}\dim(V)$. This proves the required claim. $\square$

Using this fact, we can now prove one of the main theorems of this section. Fix a maximal torus $H \subseteq G$ such that $\mathrm{Lie}(H) = \mathfrak{h}$. Let $G^{reg} \subseteq G$ be the regular, semisimple locus in G, that is, G^{reg} is the open subset of G consisting of semisimple elements with distinct eigenvalues. Define the scheme:

$$\mathfrak{X}^{reg} := \{(g, y, i) \in \mathfrak{X} : g \text{ is regular, semisimple}\}.$$

Theorem 5.1.5. *1. The scheme $\mathfrak{X}$ is irreducible and we have the equality $\overline{\mathfrak{X}^{reg}} = \mathfrak{X}$.*

2. The scheme $\mathfrak{X}$ is a reduced, complete intersection of dimension $\dim(\mathfrak{g}) + \dim(V)$.

This theorem is proven in exactly the same way as Theorem 3.1.5, with the only missing piece being provided by the following proposition:

Proposition 5.1.6. *The scheme $\mathfrak{X}^{reg}$ is irreducible.*

Proof. Let H^{reg} (resp. $\mathfrak{h}^{reg}$) denote the regular locus inside H (resp. $\mathfrak{h}$). Then, it is clear that $\mathfrak{X}^{reg} = G \times^{N_G(H)} \mathfrak{X}_0$, where:

$$\mathfrak{X}_0 := \{(g, y, i) \in \mathfrak{X} : g \in H^{reg}\}.$$

Hence, in order to show that $\mathfrak{X}^{reg}$ is irreducible, it suffices to show that the action of the Weyl group $W = N_G(H)/H$ on the irreducible components of $\mathfrak{X}_0$ is transitive. For this, let $Y := \{i \in V : \omega(i, xi) = 0 \text{ for all } x \in \mathfrak{h}^{reg}\}$. In terms of coordinates defined in §2.3, we can identify V with $\mathbb{C}^{2n}$, and in that case:

$$Y = \{(x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{C}^{2n} : x_j y_j = 0 \text{ for all } j\}.$$

We have a projection map $\mathfrak{X}_0 \to H^{reg} \times V$, that maps the triple (g, y, i) to the pair (g, i). This is an affine bundle map. Hence, to prove our claim, we need to show that W acts transitively on the 2^n irreducible components of Y. This is clear because the subgroup $(\mathbb{Z}/(2))^n \subseteq W$ acts on Y by swapping the x_j's with the y_j's. $\square$

Next, we define group-theoretic analogs of the schemes C and A. Define $\mathfrak{C}$ to be the closed (not necessarily reduced) subscheme of $T^*(G)$ whose defining ideal is given by the matrix entries of $\mathrm{Ad}g(x) - x$ for $(g, x) \in G \times \mathfrak{g} \simeq T^*(G)$. Define $\mathfrak{A}$ to be the closed (not necessarily reduced) subscheme of $T^*(G)$ whose defining ideal is generated by the 2×2 minors of the matrix $\mathrm{Ad}g(x) - x$. The diagonal action of G on the space $G \times \mathfrak{g}$ induces a G-action on both $\mathfrak{C}$ and $\mathfrak{A}$.

We can prove the following theorem about these schemes:

Theorem 5.1.7. *1. We have an isomorphism of schemes:*

$$(H \times \mathfrak{h})//W \simeq \mathfrak{C}//G \simeq \mathfrak{X}//G.$$

In particular, the scheme $\mathfrak{C}//G$ is reduced.

2. We have an isomorphism of schemes:

$$\mathfrak{X}//\{\pm 1\} \simeq \mathfrak{A}.$$

In particular, the scheme $\mathfrak{A}$ is reduced.

Proof. 1. We have natural closed embeddings:

$$(H \times \mathfrak{h})//W \hookrightarrow \mathfrak{C}//G \hookrightarrow \mathfrak{X}//G.$$

As $\mathfrak{X}$ is reduced, so is the schme $\mathfrak{X}//G$. Hence, to prove that both of these maps are actually isomorphisms, we only need to establish that they are bijections on $\mathbb{C}$-points. This is the content of Lemma 5.1.8.

2. The proof of this statement is exactly the same as that of Theorem 4.1.3 with the main tool used being the shifting trick proved in Lemma 4.1.4.

$\square$

Lemma 5.1.8. *Let* $(g, y, i) \in \mathfrak{X}$. *Then, there exists a Borel subgroup* $B \subseteq G$ *such that* $g \in B$ *and* $y \in \mathrm{Lie}(B)$. *Consequently, the* G*-orbit of this point in* $\mathfrak{X}$ *is closed if and only if* g *and* y *are semisimple and* $i = 0$.

Proof. Working with the embedding of $\mathfrak{sp}(V)$ in $\mathfrak{gl}(V)$ and of $Sp(V)$ in $GL(V)$, the condition $(g, y, i) \in \mathfrak{X}$ can be expressed as $g \cdot y \cdot g^{-1} - y + i^2 = 0$. Hence, we have $g \cdot y - y \cdot g + i^2 \cdot g = 0$. As i^2 has rank 1, so does the matrix $i^2 \cdot g$. So, by [EG02, Lemma 12.7], we have a common eigenvector $v \in V$ for g and y. Let $V' \subseteq V$ denote the complement of v with respect to the symplectic form. Then, the space $V'/\langle v \rangle$ is a symplectic vector space fixed by both g and y, and so, by induction, we get that g and y fix a Lagrangian flag in V, which proves the first assertion of the lemma. The other assertion follows from this. $\square$

Remark 5.1.9. *The above lemma provides a group-theoretic analog of Lemma 2.1 and Corollary 2.2 of [Los21].*

5.2 Quantum Hamiltonian reduction

In this section, we compute the quantum Hamiltonian reduction of the algebra $\mathcal{D}(G)$ of differential operators on the group G at the Joseph ideal $\mathcal{K} \subseteq \mathcal{U}\mathfrak{g}$ defined in §4.2.1. We first define all the objects we're interested in.

Recall that W_{2n} is the Weyl algebra generated by $2n$ variables. We had defined in section §4.2.1 an algebra homomorphism $\Theta_1 : \mathcal{U}\mathfrak{g} \to W_{2n}$ which defines an action of the Lie algebra

$\mathfrak{g}$ on W_{2n}. In this section, we refer to the same map as Σ_1 for notational convenience (as will become clear shortly), and so, we have a map $\Sigma_1 : \mathcal{U}\mathfrak{g} \to W_{2n}$.

Next, we define an action of $\mathfrak{g}$ on the space $\mathcal{D}(G)$. We have the conjugation action of the group G on itself. Differentiating this action gives rise to a map from the Lie algebra $\mathfrak{g}$ to vector fields on G. Vector fields on G correspond to derivations $\mathrm{Der}(\mathcal{O}(G))$ of the coordinate ring $\mathcal{O}(G)$. This gives rise to a Lie algebra homomorphism:

$$\Sigma_0 : \mathfrak{g} \to \mathrm{Der}(\mathcal{O}(G)) \subseteq \mathcal{D}(G).$$

Hence, this induces an algebra homomorphism $\mathcal{U}\mathfrak{g} \to \mathcal{D}(G)$, which we also denote by Σ_0. Then, this is the quantum co-moment map for the $\mathfrak{g}$-action on $\mathcal{D}(G)$, i.e., any $x \in \mathfrak{g}$ acts on $\mathcal{D}(G)$ by taking commutator with $\Sigma_0(x)$. We now define the algebras that we will be working with:

- Consider the algebra $\mathcal{D}(G) \otimes W_{2n}$. We have the diagonal $\mathfrak{g}$-action on this algebra: Given $d \otimes w \in \mathcal{D}(G) \otimes W_{2n}$ and $x \in \mathfrak{g}$, we define the action via:

  $$x \cdot (d \otimes w) := [\Sigma_0(x), d] \otimes w + d \otimes [\Sigma_1(x), w].$$

  Let $\Sigma_2 : \mathfrak{g} \to \mathcal{D}(G) \otimes W_{2n}$ be the Lie algebra homomorphism defined via $\Sigma_2 = \Sigma_0 \otimes 1 + 1 \otimes \Sigma_1$. This can be extended to get an algebra homomorphism $\Sigma_2 : \mathcal{U}\mathfrak{g} \to \mathcal{D}(G) \otimes W_{2n}$, which is the co-moment map for the above action. Then, we have the quantum Hamiltonian reduction $\Big((\mathcal{D}(G)\otimes W_{2n})/(\mathcal{D}(G)\otimes W_{2n})\cdot\Sigma_2(\mathfrak{g})\Big)^{\mathfrak{g}}$ of $\mathcal{D}(G)\otimes W_{2n}$ at the augmentation ideal $\mathcal{U}\mathfrak{g}^+$ of $\mathcal{U}\mathfrak{g}$.

- Define $\mathfrak{J} \subseteq \mathcal{D}(\mathfrak{g})$ to be the left ideal generated by the image of $\mathcal{K} \subseteq \mathcal{U}\mathfrak{g}$ under the co-moment map Σ_0, that is, $\mathfrak{J} := \mathcal{D}(\mathfrak{g}) \cdot (\Sigma_0(\ker(\Sigma_1)))$. Then, we have the algebra $\Big(\mathcal{D}(G)/\mathfrak{J}\Big)^{\mathfrak{g}}$ which is the quantum Hamiltonian reduction of the algebra $\mathcal{D}(G)$ at the ideal $\mathcal{K} \subseteq \mathcal{U}\mathfrak{g}$.

- Finally, we consider the spherical subalgebra $eH_c^{trig}e$ of the trigonometric Cherednik algebra H_c^{trig} with the parameter c given by $c = (c_{long}, c_{short}) = (-1/4, -1/2)$.

As in the rational case, we claim that there are isomorphisms:

$$\Psi^{trig} : \Big(\mathcal{D}(G)/\mathfrak{J}\Big)^{\mathfrak{g}} \longrightarrow eH_c^{trig}e,$$

$$\Phi^{trig} : \Big(\mathcal{D}(G)/\mathfrak{J}\Big)^{\mathfrak{g}} \longrightarrow \Big((\mathcal{D}(G) \otimes W_{2n})/(\mathcal{D}(G) \otimes W_{2n}) \cdot \Sigma_2(\mathfrak{g})\Big)^{\mathfrak{g}}.$$

These are established in Theorems 5.2.4 and 5.2.5. We start by stating an analog of Lemma 4.2.3 for the ideal $\mathfrak{J}$, which has exactly the same proof.

Lemma 5.2.1. *In the algebra $\mathbb{C}[G \times \mathfrak{g}]$, we have an inclusion of ideals $\widetilde{J} \subseteq \mathrm{gr}(\mathfrak{J})$, where $\widetilde{J}$ is the defining ideal of the scheme $\mathfrak{A}$ in $G \times \mathfrak{g}$.*

In order to construct the map Ψ^{trig}, we use the group-theoretic version of the 'universal' Harish-Chandra homomorphism from Section 2.4 for the same representation of $\mathfrak{g}$ as used in §4.2.1. This gives rise to an algebra homomorphism:

$$\Psi^{trig} : \mathcal{D}(G) \longrightarrow \mathcal{D}(H^{reg})^W.$$

Let Δ_G be the Laplacian operator on G. Recall from §2.5 the trigonometric Caloger-Moser differential operator L_c^{trig} for $c = (c_{long}, c_{short}) = (-1/4, -1/2)$ and let $\mathcal{C}_c^{trig}$ denote the centralizer of L_c^{trig} in $\mathcal{D}(H^{reg})^W$. Also, let $Z\mathfrak{g} := (\mathcal{U}\mathfrak{g})^{\mathfrak{g}} \subseteq \mathcal{D}(G)$ denote the subalgebra of bi-invariant differential operators on G.

Lemma 5.2.2. *1. We have that $L_c^{trig} \in \mathrm{Im}(\Psi^{trig})$.*

2. *The map* Ψ^{trig} *induces an isomorphism:*

$$Z\mathfrak{g} \xrightarrow{\sim} C_c^{trig}.$$

Proof. 1. This follows from Proposition 1.4 of [EFK95] by observing that the image of the Laplacian Δ_G under the radial parts map is equal to L_c^{trig} up to a constant factor and conjugation by $i = \sqrt{-1}$.

2. This follows by following the proof of Proposition 7.2 of [EG02], with the only added step being the observation that we have the equality $\mathrm{gr}(Z\mathfrak{g}) = \mathrm{Sym}(\mathfrak{g})^{\mathfrak{g}}$ by Harish-Chandra's isomorphism.

□

Corollary 5.2.3. *The spherical subalgebra* $eH_c^{trig}e$ *lies in the image of* Ψ^{trig}*.*

Proof. By the above lemma, we get that $\mathcal{C}_c^{trig} \subseteq \mathrm{Im}(\Psi^{trig})$. Next, the restriction of Ψ^{trig} to $\mathbb{C}[G]^{\mathfrak{g}} \subseteq \mathcal{D}(G)^{\mathfrak{g}}$ is exactly the group-theoretic Chevalley restriction map $\mathbb{C}[G]^{\mathfrak{g}} \to \mathbb{C}[H]^W \subseteq \mathcal{D}(H^{reg})$. Therefore, to prove the claim, it suffices to show that $eH_c^{trig}e$ lies in the algebra generated by $\mathcal{C}_c^{trig}$ and $\mathbb{C}[H]^W$.

Recall the algebra $\mathfrak{S}$, which was defined as the subalgebra of $\mathcal{D}(G)$ generated by the Dunkl-Heckman operators. Then, the algebra $\mathfrak{S}^W$ lies in the image of $eH_c^{trig}e$ under the Dunkl embedding. Also, as the Dunkl operators commute with the operator L_c^{trig} (by Theorem 2.5.3), we have the inclusion $\mathfrak{S}^W \subseteq \mathcal{C}_c^{trig}$.

With the filtrations defined earlier, we have the equalities $\mathrm{gr}(eH_c^{trig}e) = \mathbb{C}[H \times \mathfrak{h}]^W$ and $\mathrm{gr}(\mathfrak{S}^W) = \mathrm{Sym}(\mathfrak{h})^W \simeq \mathbb{C}[\mathfrak{h}]^W$. We will show that the algebra $eH_c^{trig}e$ is generated by $\mathfrak{S}^W$ and $\mathbb{C}[H]^W$. For this, it suffices to show that the algebra $\mathbb{C}[H \times \mathfrak{h}]^W$ is generated as a Poisson algebra by $\mathbb{C}[\mathfrak{h}]^W$ and $\mathbb{C}[H]^W$. This is proven in Proposition A.1. □

Finally, as in the rational case, we conclude that the ideal $\mathfrak{J}^{\mathfrak{g}}$ lies in the kernel of the map Ψ^{trig}.

Theorem 5.2.4. *The map Ψ^{trig} induces an isomorphism of algebras:*

$$\Big(\mathcal{D}(G)/\mathfrak{J}\Big)^{\mathfrak{g}} \simeq eH_c^{trig}e.$$

Proof. Let ϕ^{trig} be the natural isomorphism $\mathbb{C}[H \times \mathfrak{h}^*]^W \to \mathbb{C}[H \times \mathfrak{h}]^W$ obtained by identifying $\mathfrak{h}$ with $\mathfrak{h}^*$.

The proof of this theorem follows by adapting that of Theorem 4.2.7. The main argument in this case is encapsulated in the following commutative diagram:

$$\begin{tikzcd}
\mathbb{C}[H \times \mathfrak{h}^*]^W \arrow[r, "\phi^{trig}", "\sim"'] \arrow[d, "\text{Proposition 2.5.4}"', "\sim"] & \mathbb{C}[H \times \mathfrak{h}]^W \arrow[r, "\text{Theorem 5.1.7}", "\sim"'] & (\mathbb{C}[\mathfrak{A}])^{\mathfrak{g}} \arrow[d, two heads, "\text{Lemma 5.2.1}"] \\
\mathrm{gr}(eH_ce) \arrow[d, "\text{Proposition 2.5.4}"', "\sim"] & & \Big(\mathrm{gr}(\mathcal{D}(G))/\mathrm{gr}(\mathfrak{J})\Big)^{\mathfrak{g}} \arrow[d, two heads, "\text{proj}"] \\
\mathrm{gr}(\Theta(eH_c^{trig}e)) \arrow[r, hook, "\text{Cor. 5.2.3}"] & \mathrm{gr}\Big(\Psi\Big(\mathcal{D}(G)/\mathfrak{J}\Big)^{\mathfrak{g}}\Big) & \mathrm{gr}\Big(\mathcal{D}(\mathfrak{g})/\mathfrak{J}\Big)^{\mathfrak{g}} \arrow[l, two heads, "\mathrm{gr}(\Psi^{trig})"']
\end{tikzcd}.$$

□

Theorem 5.2.5. *There is an isomorphism of vector spaces:*

$$\Phi^{trig} : \mathcal{D}(G)/\mathfrak{J} \longrightarrow \Big((\mathcal{D}(G) \otimes W_{2n})/(\mathcal{D}(G) \otimes W_{2n}) \cdot \Sigma_2(\mathfrak{g})\Big)^{\{\pm 1\}}.$$

Furthermore, the map Φ^{trig} restricts to an algebra isomorphism between the respective subspaces of $\mathfrak{g}$-invariants.

The proof of this theorem follows directly from the shifting trick in Lemma 4.1.4.

Remark 5.2.6. *As in the rational case, the graded versions of the maps Ψ^{trig} and Φ^{trig} give rise to isomorphisms between the respective commutative algebras.*

Let $(\mathcal{D}(G) \otimes W_{2n})$-*mod* denote the category of finitely generated $(\mathcal{D}(G) \otimes W_{2n})$-modules.

Consider the algebra $Z\mathfrak{g} = (\mathcal{U}\mathfrak{g})^{\mathfrak{g}} \subseteq \mathcal{D}(\mathfrak{g})$ of bi-invariant differential operators on G and let $Z\mathfrak{g}^+ = Z\mathfrak{g} \cap \mathcal{U}\mathfrak{g}^+$. Finally, recall the algebra homomorphism $\Sigma_2 : \mathcal{U}\mathfrak{g} \to \mathcal{D}(G) \otimes W_{2n}$.

Definition 5.2.1. Let $\mathcal{C}^{trig}$ be the full subcategory of $(\mathcal{D}(G) \otimes W_{2n})$-*mod* whose objects are $(\mathcal{D}(G)\otimes W_{2n})$-modules with a locally nilpotent $Z\mathfrak{g}^+$-action and a locally finite $\Sigma_2(\mathcal{U}\mathfrak{g})$-action. This category $\mathcal{C}^{trig}$ will be referred to as the category of admissible $(\mathcal{D}(G) \otimes W_{2n})$-modules.

We identify the algebra $\mathcal{D}(G) \otimes W_{2n}$ with the algebra of differential operators $\mathcal{D}(G \times L)$ for a Lagrangian subspace $L \subseteq V$. Then, given any $\mathcal{D}(G) \otimes W_{2n}$-module, we can define its characteristic variety in $T^*(G \times L) \simeq G \times \mathfrak{g} \times L$.

Using the formalism of [GG06, §7] and the ideas from §4.2.4, we can prove the following statements about this category $\mathcal{C}^{trig}$ and its relation to the category $\mathcal{O}$ of the trigonometric Cherednik algebra for the parameter $c = (-1/4, -1/2)$.

Theorem 5.2.7. *1. A $\Sigma_2(\mathcal{U}\mathfrak{g})$-locally finite module $M \in \mathcal{D}(\mathfrak{g}) \otimes W_{2n}$-mod lies in the category $\mathcal{C}^{trig}$ if and only if $\mathrm{Ch}(M) \subseteq \mathfrak{X}^{nil}$. Furthermore, all the objects in $\mathcal{C}^{trig}$ are holonomic $(\mathcal{D}(G) \otimes W_{2n})$-modules.*

2. There exists an exact functor $\mathbb{H}^{trig}$:

$$\mathbb{H}^{trig} : (\mathcal{D}(G) \otimes W_{2n})\text{-}mod \longrightarrow eH_c^{trig}e\text{-}mod$$

$$M \mapsto M^{\mathfrak{g}}.$$

The full subcategory $\ker(\mathbb{H}^{trig})$ is a Serre subcategory of $(\mathcal{D}(G) \otimes W_{2n})$-mod and the functor $\mathbb{H}^{trig}$ induces an equivalence of categories:

$$(\mathcal{D}(G) \otimes W_{2n})\text{-}mod/\ker(\mathbb{H}^{trig}) \simeq eH_c^{trig}e\text{-}mod.$$

3. The functor $\mathbb{H}$ restricts to an exact functor $\mathbb{H} : \mathcal{C}^{trig} \to \mathcal{O}(eH_c^{trig}e)$. This induces an *equivalence of categories:*

$$\mathcal{C}^{trig}/\ker(\mathbb{H}^{trig}) \simeq \mathcal{O}(eH_c^{trig}e).$$

APPENDIX A

INVARIANT DIFFERENTIAL OPERATORS ON THE MAXIMAL TORUS

Let $\mathfrak{h}$ be a finite dimensional vector space over $\mathbb{C}$ and let W be the Weyl group associated with a fixed reduced root system $R \subseteq \mathfrak{h}^*$. Let $H \simeq (\mathbb{C}^\times)^n$ be an algebraic group such that $\mathrm{Lie}(H) = \mathfrak{h}$. Then, we can identify the symplectic variety $T^*(H)$ with $H \times \mathfrak{h}$, which has a diagonal action of the Weyl group W. We consider the quotient $T^*(H)//W$ whose coordinate ring $\mathbb{C}[T^*(H)//W] = \mathbb{C}[H \times \mathfrak{h}]^W$ has a natural Poisson algebra structure. This ring contains as subalgebras $\mathbb{C}[H]^W$ and $\mathbb{C}[\mathfrak{h}]^W$.

Proposition A.1. *If R is the root system of Type C, (and thus $W = (\mathbb{Z}/(2))^n \rtimes S_n$,) the algebra $\mathbb{C}[H \times \mathfrak{h}]^W$ is generated as a Poisson algebra by $\mathbb{C}[H]^W$ and $\mathbb{C}[\mathfrak{h}]^W$.*

This result is the group-theoretic analog of a result of Wallach [Wal93] who showed that the ring of invariants $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]^W$ is generated as a Poisson algebra by the algebras $\mathbb{C}[\mathfrak{h}]^W$ and $\mathbb{C}[\mathfrak{h}^*]^W$. The proof that we give here closely follows the one by Wallach.

Proof. Fix coordinates $\{y_1, y_2, \ldots, y_n\}$ on H and dual coordinates $\{x_1, x_2, \ldots, x_n\}$ on $\mathfrak{h}^*$, so that:

$$\mathbb{C}[H \times \mathfrak{h}] \simeq \mathbb{C}[x_1, \ldots, x_n, y_1^{\pm 1}, \ldots, y_n^{\pm 1}].$$

The subgroup $S_n \subseteq W$ acts by simultaneously permuting the x_i's and y_i's and $(\mathbb{Z}/(2))^n \subseteq W$ acts by sending the relevant x_i's to $-x_i$'s and y_i's to y_i^{-1}'s. The Poisson bracket $\{\cdot, \cdot\}$ on $\mathbb{C}[H \times \mathfrak{h}]^W$ is the one induced by:

$$\{x_i, x_j\} = 0, \{y_i, y_j\} = 0, \{y_i, x_j\} = \delta_{i,j} y_i$$

for all $1 \leq i, j \leq n$. In particular, the above relations imply that $\{y_i^{-1}, x_j\} = -\delta_{i,j} y_i^{-1}$.

By an averaging argument, every polynomial in $\mathbb{C}[H \times \mathfrak{h}]^W$ is in the linear span of polynomials of the form:

$$A(x_1^{r_1} y_1^{s_1} x_2^{r_2} y_2^{s_2} \dots x_k^{r_k} y_k^{s_k}) := \frac{1}{|W|} \sum_{w \in W} w(x_1)^{r_1} w(y_1)^{s_1} w(x_2)^{r_2} w(y_2)^{s_2} \dots w(x_k)^{r_k} w(y_k)^{s_k},$$

where r_i and s_i are integers and $r_i \geq 0$. We'll prove the proposition by showing that these polynomials lie in the Poisson subalgebra generated by $\mathbb{C}[H]^W$ and $\mathbb{C}[\mathfrak{h}]^W$ by induction on k. Consider the element $D := \sum_i (y_i + y_i^{-1}) = 2n \cdot A(y_1) \in \mathbb{C}[H]^W$.

We first deal with the case $k = 1$ and consider the polynomial $A(x_1^r y_1^s)$. Without loss of generality, we can assume that $s \geq 0$. Then, we have the equality:

$$(r+s)(r+s-1)\cdots(r+1)A(x_1^r y_1^s) = \{D, \{D, \{\dots, \{D, A(x_1^{r+s}\} \dots\}\},$$

where we have taken the bracket of $A(x_1^{r+s})$ with D successively s times. Note that $A(x_1^{r+s}) \in \mathbb{C}[\mathfrak{h}]^W$, and so, this proves the base case of the induction.

In general, for any k such that $2 \leq k \leq n$, we have that:

$$A(x_1^{r_1} y_1^{s_1}) \cdot A(x_2^{r_2} y_2^{s_2} \dots x_k^{r_k} y_k^{s_k}) = \frac{n-k+1}{n} A(x_1^{r_1} y_1^{s_1} x_2^{r_2} y_2^{s_2} \dots x_k^{r_k} y_k^{s_k}) + \dots,$$

where the rest of the sum on the right hand side consists of terms with a smaller k. This completes the proof by induction. □

Corollary A.2. *The ring of invariant differential operators $\mathcal{D}(H)^W$ is generated as an algebra by the subalgebra of bi-invariant differential operators and the algebra $\mathbb{C}[H]^W$ of invariant functions.*

Remark A.3. *The above proposition and corollary are also true for the root system of Type A, which can be shown by appropriately adapting the above proof.*

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