Commuting variety

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$.
Define the commuting scheme:

$$C(\mathfrak{g}) = \{(x,y) \in \mathfrak{g} \times \mathfrak{g} : [x,y]=0\}.$$

More precisely, it is the closed subscheme of $\mathfrak{g} \times \mathfrak{g}$ whose ideal is generated by the entries of $[x,y]=0$.

Open Problem: Is $C(\mathfrak{g})$ is reduced?

- Normalization is Cohen-Macaulay?
- Irreducible
- Application to Hilb. scheme of pts.
- Application to Hitchin morphisms
- Comb. applications - cf. Knut, Gi

Let G be s.t. $\mathfrak{g} = \mathrm{Lie}(G)$.

Then, $G \curvearrowright C(\mathfrak{g})$.

We can consider the cat. quotient

$$C(\mathfrak{g})/\!/G.$$

Theorem: $C(\mathfrak{g})/\!/G$ is reduced if:

1) $\mathfrak{g} = \mathfrak{gl}(V)$ GG '06, D'_, V'_
2) $\mathfrak{g} = \mathfrak{sp}(V)$ NC '21, Los '21

Henceforth, fix a symplectic vector space V and let $\mathfrak{g} = \mathfrak{sp}(V)$. $\dim V = 2n$.

Identify $\mathrm{Sym}^2 V \simeq \mathfrak{sp}(V)$ via
$\mathrm{Sym}^2 V \subseteq V \otimes V \simeq V \otimes V^* \xrightarrow{\sim} \mathfrak{gl}(V)$

Define the almost commuting scheme:
$$X = \{ (x, y, i) \in \mathfrak{g} \times \mathfrak{g} \times V : [x, y] + i^2 = 0 \}$$

Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a fixed Cartan and W the Weyl group.

Theorem:

- X is irreducible, reduced CI of dim $\dim \mathfrak{g} + \dim V$.
- We have an isomorphism of schemes:
$$(\mathfrak{h} \times \mathfrak{h})//W \xrightarrow{\sim} C//G \xrightarrow{\sim} X//G.$$
In particular, $C//G$ is reduced.

Introduce the scheme
$$A = \{ (x, y) \in \mathfrak{g} \times \mathfrak{g} : [x, y] \text{ has rank} \leq 1 \}$$

$\exists$ a projection map

$$X \longrightarrow A$$
$$(x, y, i) \longrightarrow (x, y)$$

Proposition: $X /\!/ \{\pm 1\} \xrightarrow{\sim} A$

In particular, A is reduced.

So, we have

$$C /\!/ G \xrightarrow{\sim} X /\!/ G \xrightarrow{\sim} A /\!/ G$$

The varieties $\mathfrak{g} \times \mathfrak{g}$ and $\mathfrak{g} \times \mathfrak{g} \times V$ are symplectic and the G-action on both is Hamiltonian. So, we have moment maps.

$$\mu_0 : V \longrightarrow \mathfrak{g} \qquad i \longmapsto i^2$$

$$\mu_1 : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$
$$(x, y) \longmapsto [x, y]$$

$$\mu_2 : \mathfrak{g} \times \mathfrak{g} \times V \longrightarrow \mathfrak{g}$$
$$(x, y, i) \longmapsto [x, y] + i^2$$

Then,
$$C /\!/ G = \mu_1^{-1}(0) /\!/ G$$
$$X /\!/ G = \mu_2^{-1}(0) /\!/ G$$
$$A /\!/ G = \mu_1^{-1}(\overline{\mathcal{O}}) /\!/ G.$$

where $\mathbb{O}$ is the orbit of rank 1 elements in G.

Quantization

Let $\mathcal{D}(\mathfrak{g})$ be the ring of poly. differential operators on $\mathfrak{g}$.

$\mathcal{D}(\mathfrak{g})$ has a filtration by order of diff. operators.

Then, $$gr(\mathcal{D}(\mathfrak{g})) = \mathbb{C}[T^*\mathfrak{g}] \simeq \mathbb{C}[\mathfrak{g} \times \mathfrak{g}]$$

Fix $L \subseteq V$ a Lagrangian subspace, let $W_{2n} = \mathcal{D}(L)$. Then, W_{2n} denote the Weyl algebra on $2n$ variables. Hence,

$$W_{2n} = \frac{\mathbb{C}\langle x_1, \dots, x_n, y_1, \dots, y_n\rangle}{([x_i, x_j] = [y_i, y_j] = 0, [x_i, y_j] = \delta_{ij})}$$

We have a $\mathfrak{g}$-action on both of these: We construct Lie algebra map.

$$\theta_0 : \mathfrak{g} \longrightarrow W_{2n} = \mathcal{D}(L)$$
$$\theta_1 : \mathfrak{g} \longrightarrow \mathcal{D}(\mathfrak{g})$$

$\theta_1 \to$ arises by the infinitesimal version of the adjoint G-action on $\mathfrak{g}$.

θ_0 arises via a symmetrization procedure:

$\mathfrak{g} \curvearrowright V$

gives a moment map

$$V \longrightarrow \mathfrak{g}^*$$

$\rightsquigarrow$ Lie algebra homo.

$$\begin{array}{ccc} \mathbb{C}[\mathfrak{g}] & \longrightarrow & \mathbb{C}[V] \\ & & \big\downarrow \text{Symm.} \\ & & W_{2n} \end{array}$$

$$\theta_2 : \mathfrak{g} \longrightarrow \mathcal{D}(\mathfrak{g}) \otimes W_{2n}$$

Classical setting	Quantum setting
$C//G = \mu_1^{-1}(0)//G$	$\left(\mathcal{D}(\mathfrak{g}) / \mathcal{D}(\mathfrak{g}) \cdot \theta_1(\mathfrak{g})\right)^{\mathfrak{g}}$
$\wr\|$	
$X//G = \mu_2^{-1}(0)//G$	$\left(\mathcal{D}(\mathfrak{g}) \otimes W_{2n} / (\mathcal{D}(\mathfrak{g}) \otimes W_{2n}) \cdot \theta_2(\mathfrak{g})\right)^{\mathfrak{g}}$
$\wr\|$	
$A//G = \mu_1^{-1}(\bar{\mathcal{O}})//G$	$\left(\mathcal{D}(\mathfrak{g}) / J\right)^{\mathfrak{g}}$

Theorem: (Joseph) $\exists!$ primitive ideal $I \subseteq \mathcal{U}\mathfrak{g}$ s.t. $\mathrm{gr}(I) \subseteq \mathrm{Sym}(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{g}]$ is the defining ideal of rank 1 elements.

$$J = \mathcal{D}(\mathfrak{g}) \cdot \theta_1(I)$$

Theorem: (Harish-Chandra's radial parts isomorphism) (HC, Wa, L-S)

$$\left(\mathcal{D}(\mathfrak{g}) / \mathcal{D}(\mathfrak{g}) \cdot \theta_1(\mathfrak{g})\right)^{\mathfrak{g}} \simeq \mathcal{D}(\mathfrak{h})^W$$

Prop^n: [G]

$$\left(\mathcal{D}(\mathfrak{g}) \otimes W_{2n} / (\mathcal{D}(\mathfrak{g}) \otimes W_{2n}) \cdot \theta_2(\mathfrak{g})\right)^{\pm\{1\}} \simeq \mathcal{D}(\mathfrak{g})/J$$

$$D \otimes 1 \longleftarrow\!\shortmid\; D$$

Def^n of rational Cherednik algebras:

$\mathfrak{h} \subseteq \mathfrak{g}$ $\qquad W \to$ Weyl group

Let $R \subseteq \mathfrak{h}^*$ be a root system

$c : R \longrightarrow \mathbb{C}$ be a W-invariant f^n.

In type C, such a function c can be viewed as $c = (c_{long}, c_{short}) \in \mathbb{C}^2$.

Def^n: The rational Cherednik algebra for the para. c is defined as the algebra generated by $\mathfrak{h}$, $\mathfrak{h}^*$ and $\mathbb{C}W$ s.t.

$$[x, x'] = 0 \quad , \quad [y, y'] = 0$$

$$w x w^{-1} = w(x), \quad w y w^{-1} = w(y)$$

$$[y, x] = \langle y, x\rangle - \sum_{\alpha \in R} c_\alpha \langle \alpha, y\rangle \langle \alpha^\vee, x\rangle s_\alpha$$

$\forall \; x, x' \in \mathfrak{h}^*, \; y, y' \in \mathfrak{h}, \; w \in W$

$s_\alpha \longrightarrow$ reflection relative to the root α.

Let $e = \frac{1}{|W|} \sum_{w \in W} w$.

Let $eH_ce \subseteq H_c$ be the spherical subalgebra.

$$eH_0e = \mathcal{D}(\mathfrak{h})^W.$$

Theorem: [G] $\left(\mathcal{D}(\mathfrak{g})/J\right)^{\mathfrak{g}} \simeq eH_ce$

where $c = (-1/4, -1/2)$.

Applications to representation theory:

$$X = \{(x,y,i) \in \mathfrak{g} \times \mathfrak{g} \times V : [x,y] + i^2 = 0\}$$
$$\cup\!\!\mathsf{I}$$
$$X^{nil} = \{(x,y,i) : y \text{ is nilpotent}\}.$$

Theorem: [G] X^{nil} is a Lagrangian subvariety of $\mathfrak{g} \times \mathfrak{g} \times V$.

We can identify $\mathcal{D}(\mathfrak{g}) \otimes W_{2n} = \mathcal{D}(\mathfrak{g} \times L)$.
Hence, any $\mathcal{D}(\mathfrak{g}) \otimes W_{2n}$-mod M has a
$$SS(M) \subseteq T^*(\mathfrak{g} \times L) = \mathfrak{g} \times \mathfrak{g} \times V$$

$Sym(\mathfrak{g})^{\mathfrak{g}} \subseteq \mathcal{D}(\mathfrak{g})$ const. coeff. diff. op.
$\mathcal{U}$ = subalgebra gen. by $\mathcal{D}_2(\mathfrak{g})$ and eu

Def^n: Let $\mathcal{C}$ be the full subcat. of $\mathcal{D}(\mathfrak{g}) \otimes W_{2n}$-mod M s.t. $Sym(\mathfrak{g})^{\mathfrak{g}}_+$-action is locally nilp. and $\mathcal{U}$-action is locally finite.

Prop^n: Given M with locally finite $\mathfrak{n}$-action,
$M \in \mathcal{C}$ iff $SS(M) \subseteq X^{nil}$.

$\exists$ exact functor

$$\mathbb{H} : (\mathcal{D}(\mathfrak{g}) \otimes W_{2n})\text{-mod} \longrightarrow e\mathbb{H}_c e\text{-mod}$$
$$M \longmapsto M^{\mathfrak{g}}$$

which induces an isom

$$(\mathcal{D}(\mathfrak{g}) \otimes W_{2n})\text{-mod}/\ker \mathbb{H} \xrightarrow{\sim} e\mathbb{H}_c e\text{-mod}$$

$$\mathcal{C} \subseteq \mathcal{D}(\mathfrak{g}) \otimes W_{2n}\text{-mod}$$
$$\mathcal{O}(e\mathbb{H}_c e) \subseteq e\mathbb{H}_c e\text{-mod}.$$

We get an equivalence

$$\mathcal{C}/\ker \mathbb{H} \longrightarrow \mathcal{O}(e\mathbb{H}_c e).$$