Mechanics and Hamiltonian Reduction

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Structure of the talk: - Generalities on Hamiltonian mechanics - Hamiltonian reduction - Case study: Calogno - Moser space Non-commutative algebras Quantum Hamiltonian reduction - Case study : Cheednik algebeas

Genualities on Hamiltonian mechanies

Let
$$(M, w)$$
 be a symplectic manifold
Recall that given any function $F \in C^{\infty}(M)$, we
can define an associated Hamiltonian vector
field X_F via the formula
 $i_{X_F} w = dF$

This allows us to define a Poisson bracket on $C^{\infty}(M)$:

Given
$$\xi, g \in C^{\infty}(M),$$

define $\xi \xi, g \xi := w(X_{\xi}, X_{g})$

Then, this bracket satisfies the following properties:
a)
$$\xi_{1}, g_{2}^{2} = -\xi_{2}, f_{3}^{2}$$

b) $\xi_{1}, \xi_{2}, h_{3}^{2} + \xi_{2}, \xi_{1}, f_{3}^{2}$
 $+ \xi_{1}, \xi_{1}, g_{3}^{2} = 0$
c) $\xi_{1}, g_{1} = \xi_{1}, g_{1}^{2} + g_{1}^{2} + g_{2}^{2}$

Example: If $M = IR^{2n}$ with $\omega = \sum_{i=1}^{n} dp_i \wedge dx_i$ thun, the Poisson bracket can be defined via the formulae: $\sum_{i=1}^{n} dp_i \wedge dx_i$ formulae: $\sum_{i=1}^{n} dp_i \wedge dx_i$ formulae: $\sum_{i=1}^{n} dp_i \wedge dx_i$

In general,
$$\xi F, x_i = \frac{\partial F}{\partial p_i}$$

and $\xi F, p_i = \frac{\partial F}{\partial x_i}$

In Hamiltonian mechanics, we have a symplectic
manifold M as a phase space and a special
Hamiltonian function H, that dictates the
equations of motion via the relation:
$$\frac{dy}{dt} = \xi H, y \xi \qquad \text{Hamilton's} \\ equations \\ for any $y \in C^{\infty}(M).$$$

In the previous example,

$$\frac{dx_i}{dt} = \{2H, x_i\} = \frac{2H}{2p_i}$$
and
$$\frac{dp_i}{dt} = \{2H, p_i\} = \frac{2H}{2x_i}$$

Example: Suppose
$$(M, \omega) = (\mathbb{R}^2, dp \wedge dx)$$

Let the Hamiltonian function
$$H = \frac{p^2}{2} + V(x)$$
.

$$\frac{dx}{dt} = \frac{\partial H}{\partial \beta} = \beta$$

$$\frac{d\beta}{dt} = -\frac{\partial H}{\partial x} = -\frac{dV(x)}{dx}$$
which is an expression of Newton's Law.

Hamiltonian reduction

Given a phase space M and a Hamiltonian H, we'd like to solve the Hamilton's equations One way is as follows:

Suppose there exists a group G with a symplectic action on M with moment map

p: M -> of *, where of = Lie (G). Also, suppose the G-action preserves the function H.

Let O be a co-adjoint orbit. Then, we have the Hamiltonian reduction of M at O:

 $R(M, G, O) := p^{-1}(O) //G$

Assuming G acts fuely on
$$p^{-\prime}(0)$$
,
dim $R(M, G, 0) = \dim M + \dim 0 - 2\dim (G)$
 $\leq \dim M$.

a) Solve the Hamilton's equations for H on R (M, G, O).

b) Lift these solutions to those on M.

In practice, the above steategy almost never works. In fact, the situation is guite the reverse.

Example :

 $X = \{(x_1, x_2, \dots, x_n) : x_i \neq x_j, \forall i \neq j\}$ Consider $M = T^* X_g$ with coordinates (x_i, p_i) . and let

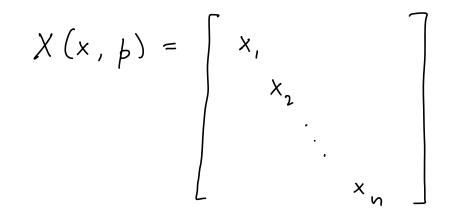
Define the Hamiltonian on M via the formula:

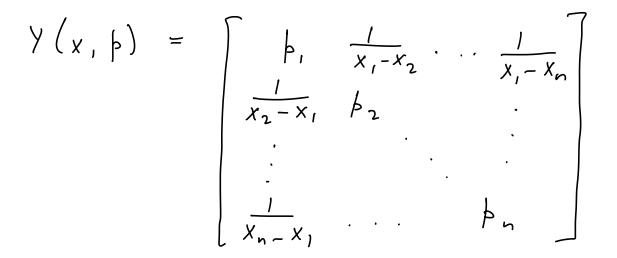
$$H = \sum_{i} p_{i}^{2} - \sum_{\substack{i \neq j}} \frac{1}{(x_{i} - x_{j})^{2}}$$

This is the phase space of a system of n particles moving along a straight line with potential x _____ dictance²

Hamilton's equations: $\frac{dx_i}{dt} = 2\beta_i , \quad \frac{d\beta_i}{dt} = \frac{2}{j\neq i} \frac{2}{(x_i - x_j)^2}$

Consider the following construction:





What is the commutator [X(x,p), Y(x,p)]?

$$\begin{bmatrix} X(x, p), Y(x, p) \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \vdots \\ \vdots & \ddots & 1 \\ \vdots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{bmatrix}$$

And so, [X(x,p), Y(x,p)] + I is a rank 1 matrix 1 matrix! Let $C_n = \mathcal{Z}(x, y) \in \mathcal{G}_n \times \mathcal{G}_n : \operatorname{rank}([x, y] + 1) = 1$ Suppose $U_n \subseteq C_n$ is the open subset containing pairs (X, Y) such that X is diagonalizable. So, we have constructed a map: $(x_1,\ldots,x_n,\beta_1,\ldots,\beta_n) \longmapsto (x(x,\beta),Y(x,\beta))$

The space
$$U_n$$
 has a natural action of
 $G = GL_n$ by conjugation:
 $g \cdot (X, Y) := (g X g^{-1}, g Y g^{-1}).$

Theorem: The above map induces an
isomosphiem:
$$\overline{\partial}: M//S_n \xrightarrow{\sim} U_n//GL_n$$

This space is known as the Calogero-
Moser space.

- · Why is this theorem relevant to solving Hamilton's equations on M?
- · How is this related to Hamiltonian reduction ?

Consider the symplectic manifold $(T^* gl_n \cong gl_n \times gl_n, \sum_{i,j} dY_{ij} \wedge dX_{ji})$

The conjugation action of $G = GL_n$ is symplectic with moment map:

$$V: gl_n \times gl_n \longrightarrow gl_n^* \cong gl_n$$

$$(X, Y) \longmapsto [X, Y]$$

Let
$$O \subseteq gl_n$$
 be the GL_n orbit consisting
of matrices M such that $M+I$ has rank 1.

Then, we have the Hamiltonian reduction

$$R(M,G,O) = p^{-1}(O)/G$$

 $= \frac{2}{2}(x,y)$: rank $([x,y]+I)=1\frac{3}{6L_n}$
 $= C_n//GL_n$.

Hence, $\simeq M//S_n$. $R(M,G,O) = C_n // GL_n$

Recall the map
$$\theta$$
:
 $\theta : M \longrightarrow U_{n}$
 $(x_{1}, \dots, x_{n}, y_{1}, \dots, y_{n}) \longmapsto (x(x, p), Y(x, p))$
where
 $Y(x, p) = \begin{bmatrix} p_{1} & \frac{1}{x_{1} - x_{2}} & \dots & \frac{1}{x_{n} - x_{n}} \\ \frac{1}{x_{n} - x_{n}} & p_{2} & \dots & \frac{1}{x_{n}} \\ \dots & \dots & \dots & \dots \\ \frac{1}{x_{n} - x_{n}} & \dots & p_{n} \end{bmatrix}$

Thun,
$$\operatorname{Tr}\left(Y(x, \beta)^{2}\right)$$

= $\underset{i}{\leq} p_{i}^{2} - \underset{i\neq j}{\leq} \frac{1}{(x_{i} - x_{j})^{2}}$

So, we consider the function $\tilde{H} = Tr(Y^2)$ on the symplectic space $gl_n \times gl_n$, with Hamilton's equation

$$\frac{dY}{dt} = \frac{\partial H}{\partial x}, \quad \frac{dX}{dt} = -\frac{\partial H}{\partial y}.$$

These equations have quite explicit
solution:
$$X(t) = X(0) + 2t Y(0)$$
$$Y(t) = Y(0)$$

This gives us information about the evolution of x; and p; over time.

Non-commutative algebras

Given an associative (not necessarily commutative algebra) A, we want to study representations of A, which are algebra homomorphisms $\phi: \mathcal{A} \longrightarrow \mathfrak{gl}(v)$ for some victor space V.

A representation is said to be irreducible if it doesn't contain a proper sub-representation.

Example: (Schur's Lemma)

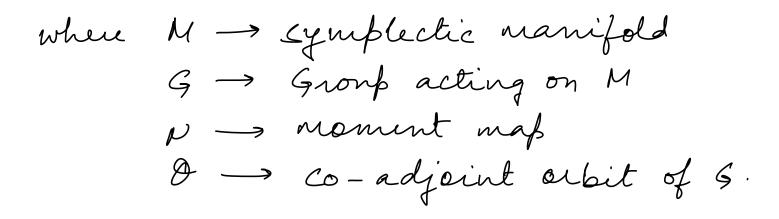
If A is commutative, we have a bijection:

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In general, can we generalize the machinery of Hamiltonian reduction to arbitrary algebras A?

In the commutative setting, we defind

 $R(M,G,O) = P^{-1}(O)//G,$



We try to think of this definition algebraically.

Given $p: M \longrightarrow q^*$, we get a map on coordinate sings: $p^*: \mathbb{C}[q^*] \longrightarrow \mathbb{C}[M]$ known as the co-moment map.

Given a co-adjoint orbit $O \subseteq q^*$, we get an ideal $I \subseteq \mathbb{C}[q^*]$ such that

$$C[O] = C[g^*]/I,$$

that is, the coordinate ring of
$$\sigma$$
 is given by $\Omega[q^*]/I$.

Then, the coordinate ring of
$$p^{-1}(0)$$
 is
given by $C[M]/p*(I)$.

We get out that
$$I \subseteq L[of]$$
 such that
 $C[O] = C[g^*]/I$,
that is, the coordinate ring of O is
given by $C[g^*]/I$.
Then, the coordinate ring of $p^{-1}(O)$ is
given by $C[M]/p^*(I)$.
=) The coordinate ring of
 $R(M, G, O) = p^{-1}(O)//G$
is given by
 $\left(\frac{C[M]}{p^*(I)}\right)^G$

The above algebraic definition of Hamiltonian reduction motivates the following:

Let \mathcal{A} be an associative algebra and Let \mathcal{G} be an algebraic group acting on \mathcal{A} . Hence, we have a group homomorphism: $\overline{\Phi}: \mathcal{G} \longrightarrow \operatorname{Aut}(\mathcal{A}).$

Differentiating this, we get a map
$$\theta$$
 of $d \notin i$ of $- \Rightarrow End(A)$.
We say that the G-action on A is supplectic if we have a Lie algebra map:
 $\theta:$ of $- A$
such that for all $x \in 0$ and $a \in A$,
 $d \notin (x)(a) = [\theta(x), a]$.

Let Noj denote the universal enveloping algebra of og . Thun, D can be extended to a map: D: Noj -> A. This map is known as the quantum Co-moment map.

Let J be any 2-sided ideal in Ug.

Def": The quantum Hamiltonian eduction of the algebra A at the ideal T is defined as the algebra: $\left(\frac{\mathcal{A}}{\mathcal{D}(\mathcal{J})\cdot\mathcal{A}}\right)^{\frac{1}{2}}$

Idea: In quantum mechanics, the phase space gets replaced by a Hilbert space on which a (non-commutative) algebra of operators acts.

Quantum Hamiltonian reductions aids the solving of the relevant Schoolinger's equation.

Hope: The process of quantum Hamiltonian enduction allows us to construct interesting, but hard to study, algebras from easier to study algebras A.

Example: Let
$$A = \frac{C \langle x, y \rangle}{(yx - xy = i)}$$
 Weyl algebra

Recall the product rule from calculus:

$$\frac{d}{dx} (x f(x)) = f(x) + x \frac{d}{dx} (f(x)).$$

As operators acting on
$$f(x)$$
,
 $\frac{d}{dx} = 1 + \frac{d}{dx}$
 $\frac{d}{dx} = 1 + \frac{d}{dx}$
 $\frac{d}{dx} = 1 + \frac{d}{dx} = 1$ or $\left[\frac{d}{dx}, x\right] = 1$

Hence, we can think of A as the algebra
$$C(x, \frac{d}{dx})$$
, which is the

ring of differential operators acting on the affine space A.

Let
$$X = /A^{\prime\prime}$$

Then, the ring of differential operators D(A") is generated by: • The coordinate functions x_i • The partial derivatives $\frac{1}{3x_i}$

We have the relations

$$\begin{bmatrix} x_{i}, x_{j} \end{bmatrix} = 0, \qquad \begin{bmatrix} \frac{\partial}{\partial x_{i}}, \frac{\partial}{\partial x_{j}} \end{bmatrix} = 0$$

$$\begin{bmatrix} \frac{\partial}{\partial x_{i}}, x_{j} \end{bmatrix} = S_{ij}$$

$$\{x_i, x_i\} = 0$$
, $\{z_i, z_i\} = 0$,
 $\{z_i, x_i\} = \{z_i\}$
 $\{z_i, x_i\} = \{z_i\}$

Hence, there is some relation between the Poisson algebra $O(\mathbb{C}^{2n})$ and the noncommutative algebra D(Cⁿ).

Keyword: $D(\mathbb{C}^n)$ is a "quantization" of $O(\mathbb{C}^{2n})$.

Thue is an algebra He that can be obtained by a quantum Hamiltonian reduction of D(gln) and which is a quantization of the Calogero - Mosu space.

quantize $\mathcal{D}(gl_n)$ Olglaxgla) -Ham. reduction quant Ham. reduction quantize He Calogno-Mosa space

Juantization commutes with reduction. Guiding philosophy:

Def": The algebra He is known as the rational Churchnik algebra.

- This algebra has applications to combinatorics and the representation theory of Hecke algebras.

- Just like we can quantize Poisson algebras, we can quantize a specifie class of representations.

Just like
$$T^*X$$
 is the prototypical example
of a symplectic space, $D(x)$ is the proto-
typical non-commutative algebra that one
studies via quantum Hamiltonian reduction.

Def": Given a filtered algebra \mathcal{A} : $\mathcal{A}_{o} \subseteq \mathcal{A}_{i} \subseteq \mathcal{A}_{2} \subseteq \ldots$, we define the associated graded algebra gr (\mathcal{A}) via: $gr (\mathcal{A}) := \bigoplus_{i=0}^{\infty} \mathcal{A}_{i+i}/\mathcal{A}_{i}$.

Example / Proposition: For any X, we have an
isomorphism of Poisson algebras:
$$g_{1}(D(X)) \simeq O(T^{*}X).$$

Thank You!