Mechanics and Hamiltonian Reduction

- Pallav Goyal

Structure of the talk:

- Generalities on Hamiltonian mechanics
- Hamiltonian reduction
    - Case study: Calogero-Moser space
- Non-commutative algebras
- Quantum Hamiltonian reduction
    - Case study: Cherednik algebras

Generalities on Hamiltonian mechanics

Let (M, ω) be a symplectic manifold.
Recall that given any function $F \in C^\infty(M)$, we can define an associated Hamiltonian vector field X_F via the formula:

$$i_{X_F} \omega = dF$$

This allows us to define a Poisson bracket on $C^\infty(M)$:

Given $f, g \in C^\infty(M)$,
define $\{f, g\} := \omega(X_f, X_g)$

Then, this bracket satisfies the following properties:

a) $\{f, g\} = -\{g, f\}$

b) $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$

c) $\{f, gh\} = \{f, g\}h + g\{f, h\}$

<u>Example</u> : If $M = \mathbb{R}^{2n}$ with $\omega = \sum_{i=1}^{n} dp_i \wedge dx_i$ then, the Poisson bracket can be defined via the formulae:

$$\boxed{\begin{gathered}\{x_i, x_j\} = 0, \quad \{p_i, p_j\} = 0 \\ \{p_i, x_j\} = \delta_{ij}.\end{gathered}}$$

In general, $\{F, x_i\} = \dfrac{\partial F}{\partial p_i}$

and $\{F, p_i\} = \dfrac{-\partial F}{\partial x_i}$

In Hamiltonian mechanics, we have a symplectic manifold M as a phase space and a special Hamiltonian function H, that dictates the equations of motion via the relation:

$$\frac{dy}{dt} = \{H, y\}$$

Hamilton's equations

for any $y \in C^\infty(M)$.

In the previous example,

$$\frac{dx_i}{dt} = \{H, x_i\} = \frac{\partial H}{\partial p_i}$$

and $$\frac{dp_i}{dt} = \{H, p_i\} = \frac{\partial H}{\partial x_i}$$

<u>Example</u> : Suppose $(M, \omega) = (\mathbb{R}^2, dp \wedge dx)$

Let the Hamiltonian function H be defined as:

$$H = \frac{p^2}{2} + V(x).$$

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = p$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x} = -\frac{dV(x)}{dx}$$

which is an expression of <u>Newton's law</u>.

Hamiltonian reduction

Given a phase space M and a Hamiltonian H, we'd like to solve the Hamilton's equations. One way is as follows:

Suppose there exists a group G with a symplectic action on M with moment map

$$\mu : M \longrightarrow \mathfrak{g}^*,$$

where $\mathfrak{g} = \mathrm{Lie}(G)$.

Also, suppose the G-action preserves the function H.

Let $\mathcal{O}$ be a co-adjoint orbit. Then, we have the Hamiltonian reduction of M at $\mathcal{O}$:

$$R(M, G, \mathcal{O}) := \mu^{-1}(\mathcal{O}) /\!/ G$$

As G preserves the Hamiltonian H, it descends to a function on $R(M, G, \theta)$.

Assuming G acts freely on $\mu^{-1}(\theta)$,

$$\underline{\dim R(M, G, \theta)} = \dim M + \dim \theta - 2 \dim (G) < \underline{\dim M}.$$

Hence, $R(M, G, \theta)$ is a symplectic manifold of strictly smaller dimension than M, and so, we can try the following:

a) Solve the Hamilton's equations for H on $R(M, G, \theta)$.

b) Lift these solutions to those on M.

In practice, the above strategy almost never works. In fact, the situation is quite the reverse.

Example:

Consider $X = \{(x_1, x_2, \ldots, x_n) : x_i \neq x_j \text{ if } i \neq j\}$ and let $M = T^*X$, with coordinates (x_i, p_i).

Define the Hamiltonian on M via the formula:

$$\boxed{H = \sum_i p_i^2 - \sum_{i \neq j} \frac{1}{(x_i - x_j)^2}}$$

This is the phase space of a system of n particles moving along a straight line with potential $\propto \dfrac{1}{\text{distance}^2}$.

Hamilton's equations:

$$\frac{dx_i}{dt} = 2p_i \quad , \quad \frac{dp_i}{dt} = \sum_{j \neq i} \frac{2}{(x_i - x_j)^3}$$

Consider the following construction:

$$X(x,p) = \begin{bmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{bmatrix}$$

$$Y(x,p) = \begin{bmatrix} p_1 & \frac{1}{x_1-x_2} & \cdots & \frac{1}{x_1-x_n} \\ \frac{1}{x_2-x_1} & p_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{1}{x_n-x_1} & \cdots & & p_n \end{bmatrix}$$

What is the commutator $[X(x,p), Y(x,p)]$?

$$[X(x,p), Y(x,p)] = \begin{bmatrix} 0 & 1 & \cdots & & 1 \\ 1 & 0 & & & \vdots \\ 1 & & \ddots & & \vdots \\ \vdots & & & \ddots & 1 \\ 1 & \cdots & & 1 & 0 \end{bmatrix}$$

And so,

$[X(x,p), Y(x,p)] + I$ is a rank 1 matrix!

Let $C_n = \{ (X,Y) \in gl_n \times gl_n : \operatorname{rank}([X,Y] + I) = 1 \}$

Suppose $U_n \subseteq C_n$ is the open subset containing pairs (X,Y) such that X is diagonalisable.

So, we have constructed a map:

$$\Phi : M \longrightarrow U_n$$
$$(x_1, \ldots, x_n, p_1, \ldots, p_n) \longmapsto (X(x,p), Y(x,p))$$

The space U_n has a natural action of $G = GL_n$ by conjugation:

$$g \cdot (X, Y) := (gXg^{-1}, gYg^{-1}).$$

The space M has a natural action of S_n by simultaneous permutation of x_i's and p_i's.

Theorem: The above map induces an isomorphism:

$$\bar{\theta} : M /\!/ S_n \xrightarrow{\sim} U_n /\!/ GL_n$$

This space is known as the Calogero-Moser space.

- Why is this theorem relevant to solving Hamilton's equations on M?

- How is this related to Hamiltonian reduction?

Consider the symplectic manifold
$\left(T^* \mathfrak{gl}_n \simeq \mathfrak{gl}_n \times \mathfrak{gl}_n \, , \quad \sum_{i,j} dY_{ij} \wedge dX_{ji}\right)$

The conjugation action of $G = GL_n$ is symplectic with moment map:

$$\begin{aligned} \mu : \mathfrak{gl}_n \times \mathfrak{gl}_n &\longrightarrow \mathfrak{gl}_n^* \simeq \mathfrak{gl}_n \\ (X, Y) &\longmapsto [X, Y] \end{aligned}$$

Let $\mathcal{O} \subseteq \mathfrak{gl}_n$ be the GL_n orbit consisting of matrices M such that $M + I$ has rank 1.

Then, we have the Hamiltonian reduction

$$\begin{aligned} R(M, G, \mathcal{O}) &= \mu^{-1}(\mathcal{O})/G \\ &= \{(X, Y) : \operatorname{rank}([X, Y] + I) = 1\} / GL_n \\ &= C_n // GL_n . \end{aligned}$$

Hence,

$$\underline{R(M, G, \mathcal{O}) = C_n // GL_n \simeq M // S_n} .$$

Recall the map θ :

$$\theta : M \longrightarrow U_n$$

$$(x_1, \ldots, x_n, y_1, \ldots, y_n) \longmapsto (X(x,p), Y(x,p))$$

where

$$Y(x,p) = \begin{bmatrix} p_1 & \frac{1}{x_1 - x_2} & \cdots & \frac{1}{x_1 - x_n} \\ \frac{1}{x_2 - x_1} & p_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{1}{x_n - x_1} & \cdots & & p_n \end{bmatrix}$$

Then, $\mathrm{Tr}(Y(x,p)^2)$

$$= \sum_i p_i^2 - \sum_{i \neq j} \frac{1}{(x_i - x_j)^2}$$

So, we consider the function $\tilde{H} = Tr(Y^2)$ on the symplectic space $gl_n \times gl_n$, with Hamilton's equations

$$\frac{dY}{dt} = \frac{\partial H}{\partial X}, \quad \frac{dX}{dt} = -\frac{\partial H}{\partial Y}.$$

These equations have quite explicit solutions:

$$\boxed{\begin{aligned} X(t) &= X(0) + 2t\,Y(0) \\ Y(t) &= Y(0) \end{aligned}}$$

This gives us information about the evolution of x_i and p_i over time.

Non-commutative algebras

Given an associative (not necessarily commutative algebra) $\mathcal{A}$, we want to study representations of $\mathcal{A}$, which are algebra homomorphisms

$$\phi : \mathcal{A} \longrightarrow gl(V)$$

for some vector space V.

A representation is said to be irreducible if it doesn't contain a proper sub-representation.

Example: (Schur's Lemma)

If $\mathcal{A}$ is commutative, we have a bijection:

$$\left\{ \begin{array}{l} \text{Irreducible} \\ \text{representations of } \mathcal{A} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Closed points} \\ \text{of } \mathrm{Spec}(\mathcal{A}) \end{array} \right\}$$

In general, can we generalise the machinery of Hamiltonian reduction to arbitrary algebras $\mathcal{A}$?

In the commutative setting, we defined

$$R(M, G, \mathcal{O}) = \mu^{-1}(\mathcal{O}) /\!/ G,$$

where $M \rightarrow$ symplectic manifold
$G \rightarrow$ Group acting on M
$\mu \rightarrow$ moment map
$\mathcal{O} \rightarrow$ co-adjoint orbit of G.

We try to think of this definition algebraically.

Given $\mu : M \longrightarrow \mathfrak{g}^*$, we get a map on coordinate rings:
$$\mu^* : \mathbb{C}[\mathfrak{g}^*] \longrightarrow \mathbb{C}[M]$$
known as the co-moment map.

Given a co-adjoint orbit $\mathcal{O} \subseteq \mathfrak{g}^*$, we get an ideal $I \subseteq \mathbb{C}[\mathfrak{g}^*]$ such that

$$\mathbb{C}[\mathcal{O}] = \mathbb{C}[\mathfrak{g}^*]/I,$$

that is, the coordinate ring of $\mathcal{O}$ is given by $\mathbb{C}[\mathfrak{g}^*]/I$.

Then, the coordinate ring of $\mu^{-1}(\mathcal{O})$ is given by $\mathbb{C}[M]/\mu^*(I)$.

$\Rightarrow$ The coordinate ring of

$$R(M, G, \mathcal{O}) = \mu^{-1}(\mathcal{O})//G$$

is given by

$$\boxed{\left(\frac{\mathbb{C}[M]}{\mu^*(I)}\right)^G}.$$

The above algebraic definition of Hamiltonian reduction motivates the following:

Let $\mathcal{A}$ be an associative algebra and let G be an algebraic group acting on $\mathcal{A}$.

Hence, we have a group homomorphism:

$$\Phi : G \longrightarrow \mathrm{Aut}(\mathcal{A}).$$

Differentiating this, we get a map

$$d\Phi : \mathfrak{g} \longrightarrow \mathrm{End}(\mathcal{A}).$$

$$\begin{tikzcd} & \mathcal{A} \arrow[d] \\ \mathfrak{g} \arrow[ur, dotted, "\theta"] \arrow[r, "d\Phi"'] & \mathrm{End}(\mathcal{A}) \end{tikzcd}$$

We say that the G-action on $\mathcal{A}$ is symplectic if we have a Lie algebra map:

$$\theta : \mathfrak{g} \longrightarrow \mathcal{A}$$

such that for all $x \in \mathfrak{g}$ and $a \in \mathcal{A}$,

$$d\Phi(x)(a) = [\theta(x), a].$$

Let $\mathcal{U}\mathfrak{g}$ denote the universal enveloping algebra of $\mathfrak{g}$. Thus, θ can be extended to a map:

$$\theta : \mathcal{U}\mathfrak{g} \longrightarrow \mathcal{A}.$$

This map is known as the quantum co-moment map.

Let J be any 2-sided ideal in $\mathcal{U}\mathfrak{g}$.

Def^n: The quantum Hamiltonian reduction of the algebra $\mathcal{A}$ at the ideal J is defined as the algebra:

$$\left(\frac{\mathcal{A}}{\theta(J)\cdot\mathcal{A}} \right)^G.$$

Idea: In quantum mechanics, the phase space gets replaced by a Hilbert space on which a (non-commutative) algebra of operators acts.

Quantum Hamiltonian reductions aids the solving of the relevant Schrödinger's equation.

Hope: The process of quantum Hamiltonian reduction allows us to construct interesting, but hard to study, algebras from easier to study algebras A.

Example: Let $\mathcal{A} = \dfrac{\mathbb{C}\langle x, y \rangle}{(yx - xy = 1)}$ <u>Weyl algebra</u>

Recall the product rule from calculus:

$$\frac{d}{dx}(x f(x)) = f(x) + x \frac{d}{dx}(f(x)).$$

As operators acting on $f(x)$,

$$\frac{d}{dx} x = 1 + x \frac{d}{dx}$$

$$\Rightarrow \frac{d}{dx} x - x \frac{d}{dx} = 1 \quad \text{or} \quad \boxed{\left[\frac{d}{dx}, x\right] = 1}$$

Hence, we can think of $\mathcal{A}$ as the algebra $\mathbb{C}\langle x, \frac{d}{dx} \rangle$, which is the ring of differential operators acting on the affine space $\mathbb{A}^1$.

Let $X = \mathbb{A}^n$

Then, the ring of differential operators $\mathcal{D}(\mathbb{A}^n)$ is generated by:

- The coordinate functions x_i
- The partial derivatives $\frac{\partial}{\partial x_i}$

for $1 \leq i \leq n$.

We have the relations

$$\boxed{[x_i, x_j] = 0, \qquad \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0 \qquad \left[\frac{\partial}{\partial x_i}, x_j\right] = \delta_{ij}}$$

This should feel suspiciously similar to the Poisson bracket relations on $\mathbb{C}^{2n}$ we saw earlier:

$$\{x_i, x_j\} = 0, \qquad \{p_i, p_j\} = 0,$$
$$\{p_i, x_j\} = \delta_{ij}$$

Hence, there is some relation between the Poisson algebra $\mathcal{O}(\mathbb{C}^{2n})$ and the non-commutative algebra $D(\mathbb{C}^n)$.

<u>Keyword</u>: $D(\mathbb{C}^n)$ is a "quantization" of $\mathcal{O}(\mathbb{C}^{2n})$.

Similarly, $D(\mathfrak{gl}_n)$ is a quantization of the Poisson algebra $\mathcal{O}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$.

There is an algebra H_c that can be obtained by a quantum Hamiltonian reduction of $D(\mathfrak{gl}_n)$ and which is a quantization of the Calogero-Moser space.

$$\begin{tikzcd}
\mathcal{O}(\mathfrak{gl}_n \times \mathfrak{gl}_n) \arrow[r, "\text{quantize}"] \arrow[d, "\substack{\text{Ham.}\\ \text{reduction}}"] & D(\mathfrak{gl}_n) \arrow[d, "\substack{\text{quant. Ham.}\\ \text{reduction}}"'] \\
\text{Calogero-Moser space} \arrow[r, "\text{quantize}"] & H_c
\end{tikzcd}$$

Guiding philosophy: Quantization commutes with reduction.

Def^n: The algebra H_c is known as the rational Cherednik algebra.

- This algebra has applications to combinatorics and the representation theory of Hecke algebras.

- Just like we can quantize Poisson algebras, we can quantize a specific class of representations.

Just like T^*X is the prototypical example of a symplectic space, $D(X)$ is the prototypical non-commutative algebra that one studies via quantum Hamiltonian reduction.

<u>Def^n</u>: Given a filtered algebra $\mathcal{A}$:

$$\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \dots,$$

we define the associated graded algebra $gr(\mathcal{A})$ via:

$$gr(\mathcal{A}) := \bigoplus_{i=0}^{\infty} \mathcal{A}_{i+1}/\mathcal{A}_i .$$

- $gr(A)$ has a naturally induced structure of a graded algebra.
- The commutator on $\mathcal{A}$ induces a Poisson bracket on $gr(\mathcal{A})$. (Fun exercise)

<u>Example / Proposition</u>: For any X, we have an isomorphism of Poisson algebras:

$$gr(D(X)) \simeq \mathcal{O}(T^*X).$$

Thank You!