

OLYMPIAD GEOMETRY

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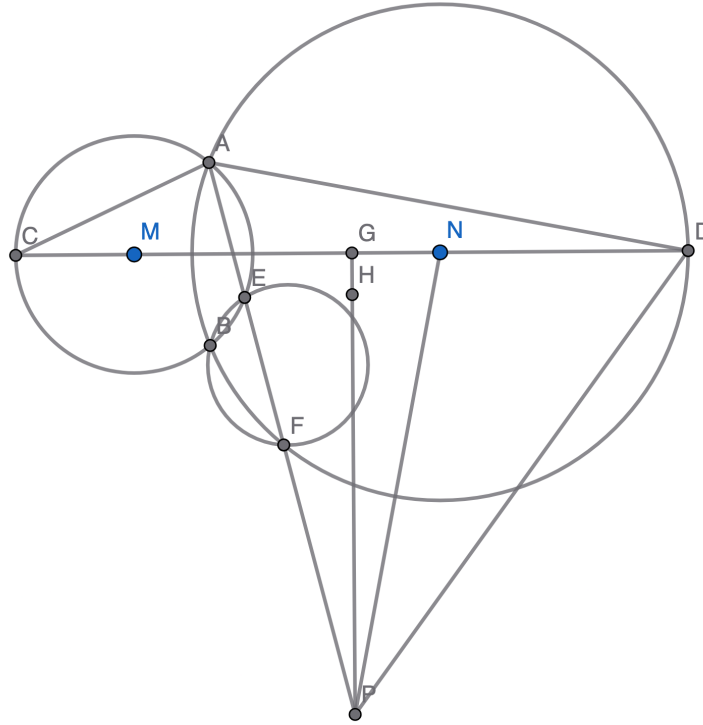
ABSTRACT. This document is a compilation of my attempts at bashing IMO geometry problems using algebraic tools.

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1. IMO 2025 Problem 2

Let Ω and Γ be circles with centres M and N , respectively, such that the radius of Ω is less than the radius of Γ . Suppose circles Ω and Γ intersect at two distinct points A and B . Line MN intersects Ω at C and Γ at D , such that points C, M, N and D lie on the line in that order. Let P be the circumcentre of triangle ACD . Line AP intersects Ω again at $E \neq A$. Line AP intersects Γ again at $F \neq A$. Let H be the orthocentre of triangle PMN . Prove that the line through H parallel to AP is tangent to the circumcircle of triangle BEF .



Solution: Suppose $AB = x$, $CD = y$, $\angle ACD = C$ and $\angle ADC = D$. Since $AB \perp CD$, it is clear that we have the following relation between these quantities:

$$y = \frac{x}{2}(\cot C + \cot D) = \frac{x \sin(C + D)}{2 \sin C \sin D}.$$

We start by computing the circumradius R of the $\triangle BEF$. To this end, we note that $\angle AEB = 180^\circ - 2C$, which implies that $\angle BEF = 2C$. Similarly, we have $\angle BFE = 2D$. Next, as P is the circumcenter of $\triangle CAD$, we have $\angle APD = 2C$. So, $\angle PAD = 90^\circ - C$. This implies that:

$$\angle BAE = \angle BAD - \angle EAD = (90^\circ - D) - (90^\circ - C) = C - D.$$

In $\triangle ABE$, by the sine rule, we have:

$$BE = AB \frac{\sin(\angle BAE)}{\sin(\angle AEB)} = x \frac{\sin(C - D)}{\sin(180^\circ - 2C)} = x \frac{\sin(C - D)}{\sin 2C}.$$

Then, the circumradius R of $\triangle BEF$ is given by:

$$R = \frac{BE}{2 \sin(\angle BFE)} = x \frac{\sin(C - D)}{2 \sin 2D \sin 2C}.$$

Next, we compute the distance of the line EF from the center of the circle BEF . Again by the sine rule, we have $EF = 2R \sin(\angle EBF) = 2R \sin(180^\circ - 2C - 2D) = 2R \sin(2C + 2D)$. Hence, the distance of EF from the center is given by:

$$\sqrt{R^2 - R^2 \sin^2(2C + 2D)} = R |\cos(2C + 2D)|.$$

Then, the distance of a tangent line to the circle BEF that is parallel to EF from the line EF is given by:

$$R \pm R |\cos(2C + 2D)| = 2R \cos^2(C + D) \text{ or } 2R \sin^2(C + D).$$

In order to show that the parallel to EF that passes through H is parallel to the circle BEF , it suffices to compute the distance of this parallel line from EF and to verify that it is equal to one of the 2 quantities above. The distance of this parallel line from EF is equal to $HP \sin(\angle HPF)$. So, to prove the required claim, it suffices to prove the equality:

$$HP \sin(\angle HPF) = 2R \cos^2(C + D).$$

Note that $\angle HPF = \angle BAE = C - D$ since $AB \parallel PG$, since both are perpendicular to CD .

Next, note that $\angle PCD = \angle PCA - C = 90^\circ - D - C$. Also, PG bisects CD and so $CG = \frac{y}{2}$. Therefore, $PG = CG \tan(\angle PCD) = \frac{y}{2} \cot(C + D)$.

Also, by the sine rule for $\triangle ABC$, we have that the radius CM of the circle ABC is equal to: $CM = \frac{AB}{2 \sin(\angle ACB)} = \frac{x}{2 \sin 2C}$. Hence,

$$\begin{aligned} MG &= CG - CM \\ &= \frac{y}{2} - \frac{x}{2 \sin 2C} \\ &= \frac{x \sin(C + D)}{4 \sin C \sin D} - \frac{x}{2 \sin 2C} \\ &= \frac{x}{4} \left(\frac{\sin(C + D) \cos C - \sin D}{\sin C \cos C \sin D} \right) \\ &= \frac{x}{4} \left(\frac{\cos(C + D) \sin C}{\sin C \cos C \sin D} \right) \\ &= \frac{x}{4} \left(\frac{\cos(C + D)}{\cos C \sin D} \right). \end{aligned}$$

Similarly, we have $NG = \frac{x}{4} \left(\frac{\cos(C + D)}{\cos D \sin C} \right)$.

Hence,

$$\tan(\angle PMG) = \frac{PG}{MG} = \frac{\frac{y}{2} \cot(C + D)}{\frac{x}{4} \left(\frac{\cos(C + D)}{\cos C \sin D} \right)} = \cot C.$$

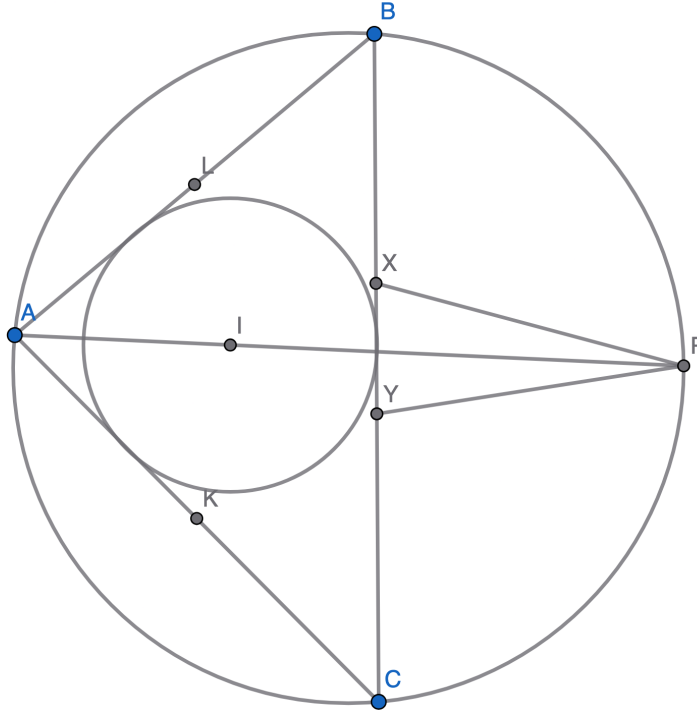
Furthermore, $\angle NHG = 180^\circ - \angle PHG = 180^\circ - (180^\circ - \angle PMG) = \angle PMG$. Therefore, $HG = NG \cot(\angle NHG) = \frac{x}{4} \left(\frac{\cos(C + D)}{\cos D \cos C} \right)$. Hence, we have:

$$\begin{aligned} HP &= PG - HG \\ &= \frac{x \cos(C + D)}{4 \sin D \sin C} - \frac{x \cos(C + D)}{4 \cos D \cos C} \\ &= \frac{x \cos^2(C + D)}{4 \sin C \sin D \cos C \cos D} \\ &= \frac{x \cos^2(C + D)}{\sin 2C \sin 2D}. \end{aligned}$$

Thus, $HP \sin(\angle HPF) = \frac{x \cos^2(C + D)}{\sin 2C \sin 2D} \sin(C - D) = 2R \cos^2(C + D)$, completing the proof.

2. IMO 2024 Problem 4

Let ABC be a triangle with $AB < AC < BC$. Let the incentre and incircle of triangle ABC be I and ω , respectively. Let X be the point on line BC different from C such that the line through X parallel to AC is tangent to ω . Similarly, let Y be the point on line BC different from B such that the line through Y parallel to AB is tangent to ω . Let AI intersect the circumcircle of triangle ABC again at $P \neq A$. Let K and L be the midpoints of AC and AB , respectively. Prove that $\angle KIL + \angle YPX = 180^\circ$.



Solution: Let $I = (0, 0)$ and the radius of the circle $\omega_1 = 1$. Let $B = (1, b)$ and $C = (1, c)$ for some $b > 0$ and $c < 0$. In order to show that $\angle KIL + \angle YPX = 180^\circ$, it suffices to show that $\tan(\angle KIL) = -\tan(\angle YPX)$.

Let the slope of the line AB be m . Then the equation of AB is:

$$\frac{y - b}{x - 1} = m.$$

As AB is tangent to ω_1 , its distance from I should be 1. Thus,

$$\frac{|m - b|}{\sqrt{1 + m^2}} = 1 \implies m = \frac{b^2 - 1}{2b}.$$

Hence, the equation of AB is:

$$\frac{y - b}{x - 1} = \frac{b^2 - 1}{2b}.$$

Similarly, the equation of AC is:

$$\frac{y - c}{x - 1} = \frac{c^2 - 1}{2c}.$$

The intersection of these lines gives the coordinates of point A :

$$A = \left(\frac{1 - bc}{1 + bc}, \frac{b + c}{1 + bc} \right).$$

Then, we compute:

$$\begin{aligned} K &= \frac{A + C}{2} = \left(\frac{1}{1 + bc}, \frac{bc^2 + 2c + b}{2(1 + bc)} \right), \\ L &= \frac{A + B}{2} = \left(\frac{1}{1 + bc}, \frac{b^2c + 2b + c}{2(1 + bc)} \right). \end{aligned}$$

Hence,

$$\begin{aligned} \text{Slope of } IK &= \frac{bc^2 + 2c + b}{2}, \\ \text{Slope of } IL &= \frac{b^2c + 2b + c}{2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \tan(\angle KIL) &= \frac{\frac{bc^2 + 2c + b}{2} - \frac{b^2c + 2b + c}{2}}{1 + \left(\frac{bc^2 + 2c + b}{2} \right) \left(\frac{b^2c + 2b + c}{2} \right)} \\ &= \frac{2(c - b)(bc + 1)}{4 + (bc^2 + 2c + b)(b^2c + 2b + c)}. \end{aligned}$$

Next, we compute the coordinates of P . As AI bisects $\angle BAC$, we have that $PB = PC$. Thus, $P = (r, \frac{b+c}{2})$ for some r . Also, since A, I and P are colinear, we have:

$$\text{Slope of } AI = \text{Slope of } PI \implies \frac{1 - bc}{b + c} = \frac{2r}{b + c}.$$

This gives that $r = \frac{1 - bc}{2}$ implying that $P = (\frac{1 - bc}{2}, \frac{b+c}{2})$.

Finally, we find coordinates of X and Y . The slope of the line through X tangent to ω is the same as the slope of AC , which is equal to $\frac{c^2 - 1}{2c}$. Hence, the equation of the tangent line is

$$y - \frac{c^2 - 1}{2c}x = \alpha$$

for some α . For this to be tangent to ω_1 , its distance from I should be 1. Therefore,

$$\frac{|\alpha|}{\sqrt{1 + \left(\frac{c^2 - 1}{2c} \right)^2}} = 1 \implies \alpha = \pm \frac{c^2 + 1}{2c}.$$

Thus, the equation of the tangent to ω through the point X is:

$$y - \frac{c^2 - 1}{2c}x = -\frac{c^2 + 1}{2c}.$$

(We choose the negative sign, since the positive sign corresponds to the line AC .) Intersecting this tangent line with the line BC , which is given by $x = 1$, we get that $X = (1, -\frac{1}{c})$. Similarly, we get that $Y = (1, -\frac{1}{b})$.

So, we can compute:

$$\begin{aligned} \text{Slope of } PX &= \frac{\frac{b+c}{2} + \frac{1}{c}}{\frac{1-bc}{2} - 1} = -\frac{bc + c^2 + 2}{c(1 + bc)}, \\ \text{Slope of } PY &= \frac{\frac{b+c}{2} + \frac{1}{b}}{\frac{1-bc}{2} - 1} = -\frac{bc + b^2 + 2}{b(1 + bc)}. \end{aligned}$$

Hence,

$$\begin{aligned}
 \tan(\angle YPX) &= \frac{-\frac{bc+b^2+2}{b(1+bc)} + \frac{bc+c^2+2}{c(1+bc)}}{1 + \left(\frac{bc+b^2+2}{b(1+bc)}\right)\left(\frac{bc+c^2+2}{c(1+bc)}\right)} \\
 &= \frac{-bc(b+c)(1+bc) - 2c(1+bc) + bc(b+c)(1+bc) + 2b(1+bc)}{bc(1+bc)^2 + (bc+b^2+2)(bc+c^2+2)} \\
 &= \frac{2(b-c)(1+bc)}{bc(1+bc)^2 + (bc+b^2+2)(bc+c^2+2)}.
 \end{aligned}$$

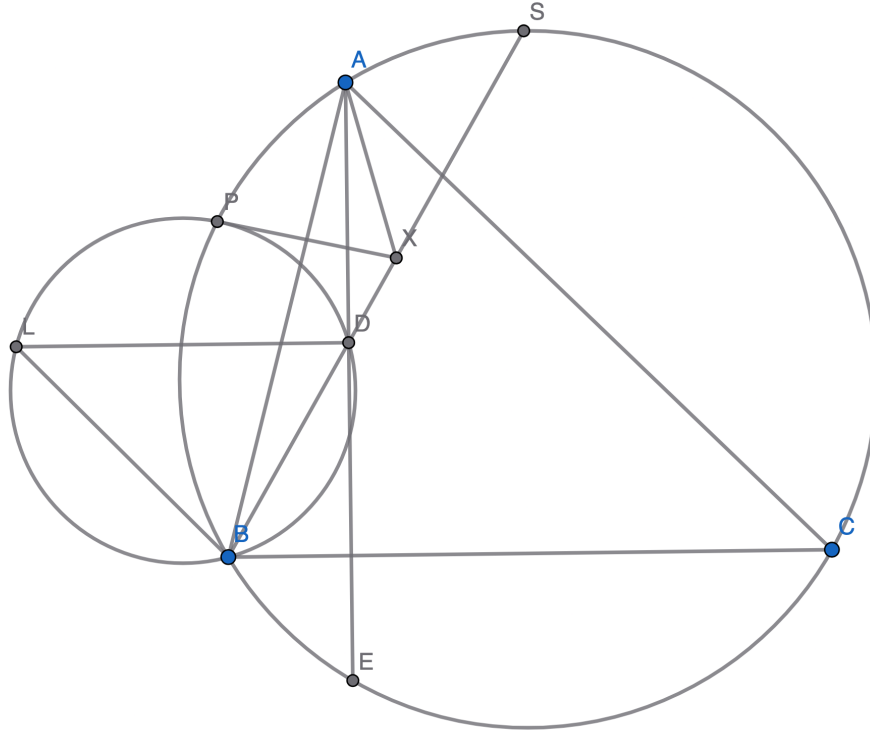
Therefore, from our expressions for $\tan(\angle KIL)$ and $\tan(\angle YPX)$, it follows that the equality $\tan(\angle KIL) = -\tan(\angle YPX)$ is equivalent to the following algebraic identity:

$$4 + (bc^2 + 2c + b)(b^2c + 2b + c) = bc(1+bc)^2 + (bc+b^2+2)(bc+c^2+2),$$

which is easily verified.

3. IMO 2023 Problem 2

Let ABC be an acute-angled triangle with $AB < AC$. Let Ω be the circumcircle of ABC . Let S be the midpoint of the arc CB of Ω containing A . The perpendicular from A to BC meets BS at D and meets Ω again at $E \neq A$. The line through D parallel to BC meets line BE at L . Denote the circumcircle of triangle BDL by ω . Let ω meet Ω again at $P \neq B$. Prove that the line tangent to ω at P meets line BS on the internal angle bisector of $\angle BAC$.



Solution: We have $\angle SBC = \angle SCB = 90^\circ - \frac{A}{2}$. Also, $\angle CBE = \angle CAE = 90^\circ - C$. Thus, $\angle LBD = 180^\circ - \angle SBE = 180^\circ - (\angle SBC + \angle CBE) = 90^\circ + \frac{C-B}{2}$.

Let $\angle PBS = t$. In ω , sine rule gives:

$$\begin{aligned} \frac{PD}{\sin t} &= \frac{LD}{\sin(\angle LBD)} \\ &= \frac{LD}{\sin(90^\circ + \frac{C-B}{2})} \\ &= \frac{LD}{\cos(\frac{B-C}{2})}. \end{aligned}$$

Next, we have $(\angle PAD + \angle DAC) + \angle PBC = 180^\circ$. Thus,

$$\begin{aligned} \angle PAD &= 180^\circ - \angle PBC - \angle DAC \\ &= 180^\circ - (t + \frac{B+C}{2}) - (90^\circ - C) \end{aligned}$$

$$= 90^\circ - t + \frac{C - B}{2}.$$

Similarly, we have $(\angle DPA + \angle DPB) + \angle ACB = 180^\circ$, Thus,

$$\begin{aligned}\angle DPA &= 180^\circ - \angle DPB - \angle ACB \\ &= 180^\circ - \angle DLB - C \\ &= 180^\circ - (90^\circ - \angle LED) - C \\ &= 180^\circ - (90^\circ - C) - C \\ &= 90^\circ.\end{aligned}$$

Hence, we have that $PD = AD \sin(\angle PAD) = AD \sin(90^\circ - t + \frac{C-B}{2}) = AD \cos(\frac{C-B}{2} - t)$. Inserting this into the above sine rule equation, we get:

$$\begin{aligned}\frac{AD \cos(\frac{C-B}{2} - t)}{\sin t} &= \frac{LD}{\cos(\frac{B-C}{2})} \\ \Rightarrow AD \cos\left(\frac{C-B}{2}\right) \cot t + AD \sin\left(\frac{C-B}{2}\right) &= \frac{LD}{\cos(\frac{B-C}{2})} \\ \Rightarrow \cot t &= \frac{LD}{AD \cos^2(\frac{B-C}{2})} + \tan\left(\frac{B-C}{2}\right).\end{aligned}$$

- (1) First, suppose PX is tangent to the circle ω . Then, $\angle DPX = \angle DBP = t$. So, by the sine rule in $\triangle PDX$ we get:

$$\begin{aligned}\frac{DX}{\sin t} &= \frac{PX}{\sin PDX} \\ &= \frac{PX}{\sin PLB}.\end{aligned}$$

Now, $\angle PLB = \angle PLD + \angle BLD = \angle PBD + \angle BLD = 90^\circ + t - C$. Hence,

$$\frac{PX}{DX} = \frac{\sin(90^\circ + t - C)}{\sin t} = \frac{\cos(t - C)}{\sin t} = \cot t \cos C + \sin C.$$

Computing the power of the point X with respect to ω , we get $PX^2 = BX \cdot DX$. Hence,

$$\frac{BX}{DX} = \frac{BX}{PX} \cdot \frac{PX}{DX} = \left(\frac{PX}{DX}\right)^2 = (\cot t \cos C + \sin C)^2.$$

- (2) Now, suppose AX bisects $\angle BAC$. Then, $\angle AXB = 180^\circ - \angle XAB - \angle XBA = 180^\circ - \frac{A}{2} - (B - \frac{B+C}{2}) = 90^\circ + C$. Also, $\angle DAX = \angle DAC - \angle XAC = 90^\circ - C - \frac{A}{2}$. So, applying the sine rule in $\triangle ADX$, we get:

$$\begin{aligned}\frac{DX}{\sin(\angle DAX)} &= \frac{AD}{\sin(\angle AXD)} \\ \Rightarrow DX &= AD \frac{\sin(90^\circ - C - \frac{A}{2})}{\sin(90^\circ + C)} = AD \frac{\cos(C + \frac{A}{2})}{\cos C}.\end{aligned}$$

Next, applying the sine rule in $\triangle BAX$, we get:

$$\begin{aligned}\frac{BX}{\sin(\angle BAX)} &= \frac{BA}{\sin(\angle AXB)} \\ \Rightarrow BX &= AB \frac{\sin(\frac{A}{2})}{\sin(90^\circ + C)} = \frac{c \sin(\frac{A}{2})}{\cos C},\end{aligned}$$

where we suppose $AB = c$. Combining the two expressions above, we get that:

$$\frac{BX}{DX} = \frac{c \sin(\frac{A}{2})}{AD \cos(C + \frac{A}{2})}.$$

We have obtained expressions for $\frac{BX}{DX}$ in both of the above cases. Thus, to prove the required claim, it suffices to prove the equality:

$$(\cot t \cos C + \sin C)^2 = \frac{c \sin(\frac{A}{2})}{AD \cos(C + \frac{A}{2})}.$$

To that end, we first compute AD . Note that $\angle ADB = 180^\circ - \angle ABD - \angle BAD = 180^\circ - (B - \frac{B+C}{2}) - (90^\circ - B) = 180^\circ - \frac{A}{2}$. Then, using the sine rule in $\triangle BAD$, we get:

$$AD = AB \frac{\sin(\angle ABD)}{\sin(\angle ADB)} = \frac{c \sin(\frac{B-C}{2})}{\sin(180^\circ - \frac{A}{2})} = \frac{c \sin(\frac{B-C}{2})}{\sin(\frac{A}{2})}.$$

This implies that:

$$\frac{c \sin(\frac{A}{2})}{AD \cos(C + \frac{A}{2})} = \frac{\sin^2(\frac{A}{2})}{\cos(C + \frac{A}{2}) \sin(\frac{B-C}{2})} = \frac{\sin^2(\frac{A}{2})}{\sin^2(\frac{B-C}{2})}$$

since $\frac{B-C}{2} = 90^\circ - (C + \frac{A}{2})$. Thus, we are reduced to proving the equality:

$$\cot t \cos C + \sin C = \frac{\sin(\frac{A}{2})}{\sin(\frac{B-C}{2})}.$$

Recall that $\cot t = \frac{LD}{AD \cos^2(\frac{B-C}{2})} + \tan(\frac{B-C}{2})$. To compute this, we need to find LD . Note that $LD = DE \tan(\angle LED) = DE \tan C = (AE - AD) \tan C$. Applying the sine rule in circle Ω , we get:

$$AE = AB \frac{\sin(\angle ABE)}{\sin(\angle ACB)} = \frac{c \sin(90 + B - C)}{\sin C} = \frac{c \cos(B - C)}{\sin C} = c \cot C \cos B + c \sin B.$$

Therefore,

$$\begin{aligned} LD &= \tan C(AE - AD) \\ &= c \tan C \left(\cot C \cos B + \sin B - \frac{\sin(\frac{B-C}{2})}{\sin(\frac{A}{2})} \right) \\ &= c \cos B + c \tan C \sin B - c \frac{\tan C \sin(\frac{B-C}{2})}{\sin(\frac{A}{2})}. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{LD}{AD \cos^2(\frac{B-C}{2})} &= \frac{\cos B + \tan C \sin B - \frac{\tan C \sin(\frac{B-C}{2})}{\sin(\frac{A}{2})}}{\frac{\sin(\frac{B-C}{2})}{\sin(\frac{A}{2})} \cos^2(\frac{B-C}{2})} \\ &= \frac{\sin(\frac{A}{2})(\cos B + \tan C \sin B) - \tan C \sin(\frac{B-C}{2})}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})}. \end{aligned}$$

This implies that:

$$\begin{aligned} \cot t &= \frac{LD}{AD \cos^2(\frac{B-C}{2})} + \tan(\frac{B-C}{2}) \\ &= \frac{\sin(\frac{A}{2})(\cos B + \tan C \sin B) - \tan C \sin(\frac{B-C}{2}) + \sin^2(\frac{B-C}{2}) \cos(\frac{B-C}{2})}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})}. \end{aligned}$$

Next,

$$\begin{aligned} \cos C \cot t &= \frac{\sin(\frac{A}{2})(\cos B \cos C + \sin C \sin B) - \sin C \sin(\frac{B-C}{2}) + \cos C \sin^2(\frac{B-C}{2}) \cos(\frac{B-C}{2})}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})} \\ &= \frac{\sin(\frac{A}{2}) \cos(B - C) - \sin C \sin(\frac{B-C}{2}) + \cos C \sin^2(\frac{B-C}{2}) \cos(\frac{B-C}{2})}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})}. \end{aligned}$$

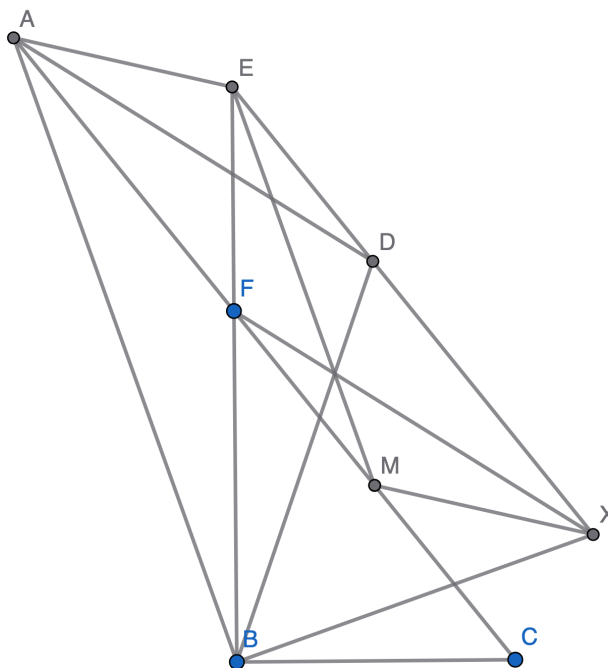
Finally,

$$\begin{aligned}
\sin C + \cos C \cot t &= \frac{\sin C \sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2}) + \sin(\frac{A}{2}) \cos(B-C) - \sin C \sin(\frac{B-C}{2}) + \cos C \sin^2(\frac{B-C}{2}) \cos(\frac{B-C}{2})}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})} \\
&= \frac{\sin(\frac{A}{2}) \cos(B-C) - \sin C \sin^3(\frac{B-C}{2}) + \cos C \sin^2(\frac{B-C}{2}) \cos(\frac{B-C}{2})}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})} \\
&= \frac{\sin(\frac{A}{2}) \cos(B-C) + \sin^2(\frac{B-C}{2}) \cos(C + \frac{B-C}{2})}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})} \\
&= \frac{\sin(\frac{A}{2}) \cos(B-C) + \sin^2(\frac{B-C}{2}) \cos(\frac{B+C}{2})}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})} \\
&= \frac{\sin(\frac{A}{2}) \cos(B-C) + \sin(\frac{A}{2}) \sin^2(\frac{B-C}{2})}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})} \\
&= \frac{\sin(\frac{A}{2}) (\cos(B-C) + \sin^2(\frac{B-C}{2}))}{\sin(\frac{B-C}{2}) \cos^2(\frac{B-C}{2})} \\
&= \frac{\sin(\frac{A}{2})}{\sin(\frac{B-C}{2})},
\end{aligned}$$

completing the proof.

4. IMO 2016 Problem 1

$\triangle BCF$ has a right angle at B . Let A be the point on line CF such that $FA = FB$ and F lies between A and C . Point D is chosen such that $DA = DC$ and AC is the bisector $\angle DAB$. Point E is chosen such that $EA = ED$ and AD is the bisector of $\angle EAC$. Let M be the midpoint of CF . Let X be the point such that $AMXE$ is a parallelogram (where $AM \parallel EX$ and $AE \parallel MX$). Prove that BD , FX and ME are concurrent.



Solution: Let $\angle BFC = y$. Also, let $BF = c$ and $FC = b$. Since $AF = FB$, we have $\angle FBA = \angle BAF = \angle FAD = \angle DAE = \frac{y}{2}$.

We start by showing that the point D lies on the EX . To prove this, it suffices to show that:

$$\frac{AD}{\sin(\angle AEX)} = \frac{DE}{\sin(\angle DAE)}.$$

Now, $AD = \frac{1}{2}AC \sec(\angle DAF) = \frac{b+c}{2} \sec(\frac{y}{2})$. Next, $\angle AEX = 180^\circ - \angle FAE = 180^\circ - y$. Finally, $DE = AE = \frac{1}{2}AD \sec(\angle DAE) = \frac{b+c}{4} \sec^2(\frac{y}{2}) = \frac{b+c}{4 \cos^2(\frac{y}{2})} = \frac{b+c}{2(\cos y + 1)} = \frac{b}{2}$. Then, we see that:

$$\frac{AD}{\sin(\angle AEX)} = \frac{\frac{b+c}{2} \sec(\frac{y}{2})}{\sin(y)} = \frac{b+c}{4 \sin(\frac{y}{2}) \cos^2(\frac{y}{2})} = \frac{DE}{\sin(\frac{y}{2})} = \frac{DE}{\sin(\angle DAE)},$$

proving that D lies on EX .

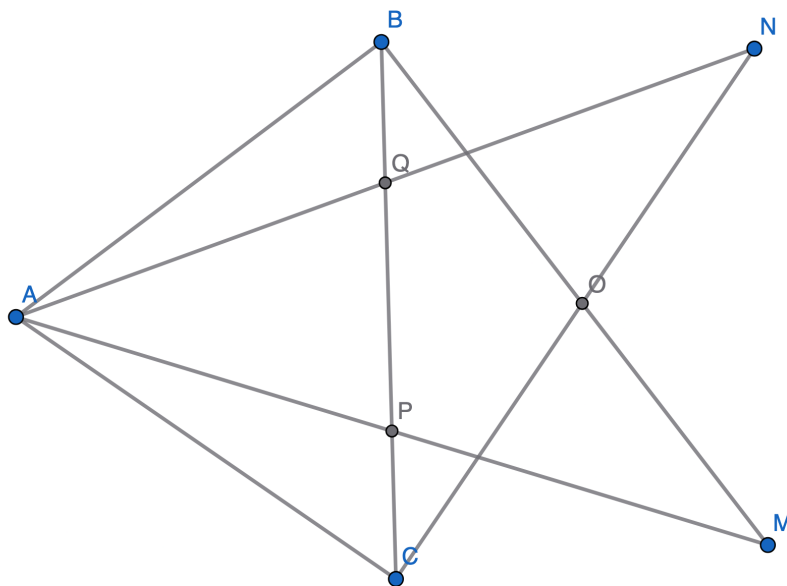
Next, we show that F lies on BE . For this, it suffices to show that $y = \angle BFC = \angle AFE$. Since we already know that $\angle FAE = y$, this is equivalent to showing that $AE = EF$. For this, we use the cosine rule in $\triangle FAE$ to observe that $FE^2 = AE^2 + AF^2 - 2AE \cdot AF \cos(\angle FAE)$. So, it suffices

to show that $AF = 2AE \cos(\angle FAE)$. Note that $AF = c$ whereas $2AE \cos(\angle FAE) = b \cos y = c$, completing the proof.

Next, we note that $EX = AM = AF + FM = c + \frac{b}{2}$. On the other hand, $EB = EF + FB = \frac{b}{2} + c$, showing that $EB = EX$. Since we already know that $EF = EA = ED$, this also implies that $DX = FB$. Hence, applying Ceva's theorem to $\triangle EBX$, the the lines EM, BD and FX will be concurrent if and only if EM bisects the side BX , which is equivalent to EM bisecting $\angle BEX$, since the triangle is isosceles. To that end, we observe that $MB = \frac{b}{2} = AE - MX$. Hence, $\triangle EMB \cong \triangle EMX$, which proves that EM is the angle bisector, thus proving the claim.

5. IMO 2014 Problem 4

Points P and Q lie on side BC of acute-angled triangle ABC so that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points M and N lie on lines AP and AQ , respectively, such that P is the midpoint of AM , and Q is the midpoint of AN . Prove that lines BM and CN intersect on the circumcircle of triangle ABC .



Solution: Note that $\angle AQP = \angle APQ = \angle CAB$. Thus $AQ = AP$. We establish a coordinate system such that $A = (0, 0)$, $B = (1, b)$, $C = (1, c)$, $Q = (1, q)$ and $P = (1, -q)$ with $b, q > 0$ and $c < 0$. Since $\angle CAQ = \angle ABC$, we must have:

$$\frac{q - c}{1 + qc} = \frac{1}{b} \implies bq - bc - qc = 1 \implies q = \frac{1 + bc}{b - c}.$$

The equation of the circumcircle ABC is given by:

$$x^2 + y^2 - (1 - bc)x - (b + c)y = 0.$$

Next, we have that $N = 2Q = (2, 2q)$ and $M = 2P = (2, -2q)$. Then we have the equation of BM :

$$\frac{y - b}{x - 1} = \frac{b + 2q}{-1} \implies y + (b + 2q)x = 2b + 2q.$$

Similarly, the equation of CN is:

$$y + (c - 2q)x = 2c - 2q.$$

The angle between these two lines is the arctan of:

$$\frac{(b+2q)-(c-2q)}{1+(b+2q)(c-2q)} = \frac{b-c+4q}{1+bc+2cq-2bq-4q^2} = \frac{b-c+4q}{-1-bc-4q^2}.$$

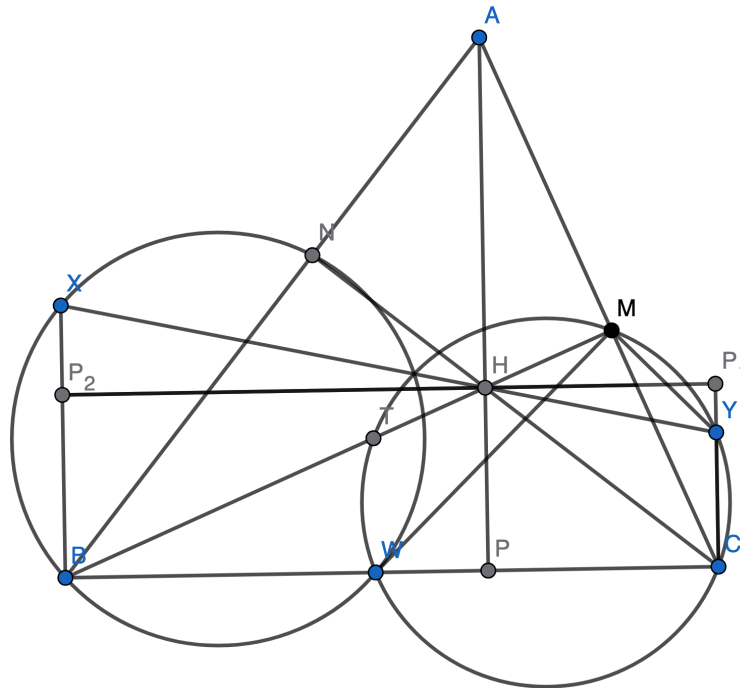
To show that A, B, O and C are concyclic, it suffices to show that this is equal to $-\tan(\angle BAC)$, that is:

$$\begin{aligned} \frac{b-c+4q}{-1-bc-4q^2} &= -\frac{b-c}{1+bc} \\ \iff 4(1+bc)q &= 4q^2(b-c) \\ \iff q &= \frac{1+bc}{b-c}, \end{aligned}$$

which was already shown, thus completing the proof.

6. IMO 2013 Problem 4

Let ABC be an acute-angled triangle with orthocentre H , and let W be a point on the side BC , lying strictly between B and C . The points M and N are the feet of the altitudes from B and C , respectively. Denote by ω_1 the circumcircle of BWN , and let X be the point on ω_1 such that WX is a diameter of ω_1 . Analogously, denote by ω_2 the circumcircle of CWM , and let Y be the point on ω_2 such that WY is a diameter of ω_2 . Prove that X , Y and H are collinear.



Solution: Let $a = BC$, $b = CA$ and $c = AB$. Let R be the circumradius of $\triangle ABC$. Construct P_1P_2 parallel to BC through the point H . To prove the required claim, it suffices to show that $\angle XHP_2 = \angle YHP_1$.

We have $\angle HMC = \angle WMY = 90^\circ$. Therefore, $\angle TMW = \angle CMY$, which implies that $TW = CY$. Consider the power of the point B with respect to the circle ω_2 :

$$BT \cdot BM = BW \cdot BC.$$

$$\begin{aligned} BT &= \frac{BW \cdot BC}{BM} \\ &= \frac{aBW}{a \sin C} = \frac{BW}{\sin C}. \end{aligned}$$

Next, note that $\angle TBW = 90^\circ - C$. Applying cosine law to $\triangle BWT$:

$$\begin{aligned} TW^2 &= BT^2 + BW^2 - 2BT \cdot BW \cos(TBW) \\ &= BT^2 + BW^2 - 2BT \cdot BW \sin C \end{aligned}$$

$$\begin{aligned}
&= \frac{BW^2}{\sin^2 C} + BW^2 - 2 \frac{BW}{\sin C} \cdot BW \sin C \\
&= BW^2 \csc^2 C - BW^2 \\
&= BW^2 \cot^2 C.
\end{aligned}$$

Thus, $TW = BW \cot C$. This implies that $CY = BW \cot C$. Similarly, we get that $BX = CW \cot B$.

As $XB \perp BC$, we must have $XB \perp P_1P_2$ since $P_1P_2 \parallel BC$. Similarly, $YC \perp P_1P_2$. Therefore,

$$\begin{aligned}
\tan(\angle XHP_2) &= \frac{XP_2}{HP_2} \\
&= \frac{BX - BP_2}{HP_2} \\
&= \frac{BX - HP}{BP} \\
&= \frac{CW \cot B - 2R \cos B \cos C}{2R \sin C \cos B} \\
&= \frac{CW}{2R \sin B \sin C} - \cot C.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\tan(\angle YHP_1) &= \frac{YP_1}{HP_1} \\
&= \cot B - \frac{BW}{2R \sin B \sin C}.
\end{aligned}$$

Then, we see that:

$$\begin{aligned}
\tan(\angle XHP_2) - \tan(\angle YHP_1) &= \frac{BW + CW}{2R \sin B \sin C} - \cot B - \cot C \\
&= \frac{a}{2R \sin B \sin C} - \frac{\sin(B+C)}{\sin B \sin C} \\
&= \frac{\sin A}{\sin B \sin C} - \frac{\sin A}{\sin B \sin C} \\
&= 0,
\end{aligned}$$

which shows that $\angle XHP_2 = \angle YHP_1$, completing the proof.