Recurrence relations

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1. Find general solutions for the following recurrence relations:

   - $a_{n+2} - a_{n+1} - 6a_n = 0$
   - $a_{n+2} - a_{n+1} - a_n = 0$
   - $a_{n+3} - 2a_{n+2} - a_{n+1} + 2a_n = 0$
   - $a_{n+4} - 3a_{n+2} + 2a_n = 0$
   - $a_{n+3} + a_{n+1} - 2a_n = 0$
   - $a_{n+2} - 5a_{n+1} + 4a_n = n^2 + n + 1$
   - $a_{n+2} - 4a_n = 3.5^n$
   - $a_{n+2} - 6a_{n+1} - 7a_n = 7^n$

2. - Given a positive integer n, find the number of ways of covering an $n \times 2$ rectangular floor with 1×2 tiles.
   - Given a positive integer n, find the number of ways of covering an $n \times 2$ rectangular floor with 1×2 and 2×2 tiles.

3. For any n, find the number of sequences consisting of 1, 2, . . . , n each exactly once such that except the leftmost number, every number in the sequence differs from some number on the left by $+1$ or -1.

4. Let $S = \{0, 1, 2\}$ and n be a positive integer.

   - Find the number of strings of length n consisting of alphabet from S such that there are no consecutive 0's and 1 never follows 0 immediately.
   - Find the number of strings of length n consisting of alphabet from S such that 1 never follows 0 immediately.

5. Fix $n \geq 2$. Suppose n people line up to board an aeroplane, but the first person in the line loses his ticket and doesn't know the seat number allotted to him. So, on going inside, he sits on a random seat. Each subsequent passenger either takes his own seat if still available, otherwise a random unoccupied seat. Find the probability that the nth person will be able to sit in his allotted seat.

6. Suppose two players A and B are playing a game. Suppose A has 37 balls and B has 63 balls. At each step of the game, if A has a balls, he gives i balls to B where i is chosen randomly between 0 and a. Similarly, if B has b balls, he gives j balls to A where j is chosen randomly between 0 and b. The game goes on till one player gets all the balls. Find the probability that A wins.

7. (IMOTC 2013, TST 1) Let $n \geq 2$ be an integer. There are n beads numbered $1, 2, \ldots, n$. Two necklaces made out of some of these beads are considered the same if we can get one by rotating the other (with no flipping allowed). For example, with $n \geq 5$, the necklace with four beads $1, 5, 3, 2$ in the clockwise order is same as the one with $5, 3, 2, 1$ in the clockwise order, but is different from the one with $1, 2, 3, 5$ in the clockwise order. We denote by $D_0(n)$ (respectively $D_1(n)$) the number of ways in which we can use all the beads to make an even number (resp. an odd number) of necklaces each of length at least 3. Prove that $n-1$ divides $D_1(n) - D_0(n)$.

8. (IMC 2004) Let S be a set of $\binom{2n}{n} + 1$ real numbers, where n is an positive integer. Prove that there exists a monotone sequence $\{a_i\}_{1 \leq i \leq n+2} \subseteq S$ such that $|x_{i+1} - x_1| \geq 2|x_i - x_1|$ for all $i = 2, 3, \ldots, n+1$.

9. Let $S = \{a, b, c\}$ and n a positive integer. Let A_n denote the set of words of length n with letters from S that do not contain two consecutive a's and do not contain two consecutive b's. Let B_n denote the number of words of length n with letters from S that do not contain three consecutive distinct letters. Show that $|B_{n+1}| = 3|A_n|$

10. (IMO 2011) Let $n > 0$ be an integer. We are given a balance and n weights of weight 2^0, 2^1, . . . , 2^{n-1}. In a sequence of n moves we place all weights on the balance. In the first move we choose a weight and put it on the left pan. In each of the following moves we choose one of the remaining weights and we add it either to the left or to the right pan. Compute the number of ways in which we can perform these n moves in such a way that the right pan is never heavier than the left pan