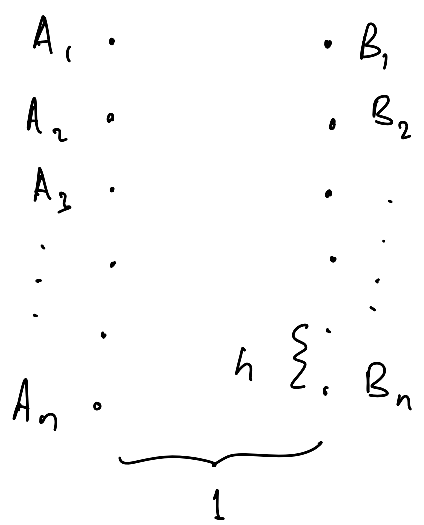
Mathematics of Shoelacing

Framework

Def^n: A mathematical shoe of stretch h with $2n$ eyelets:

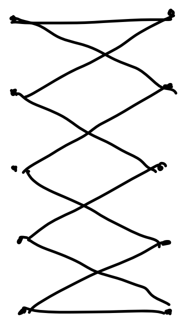


$n > 2$.

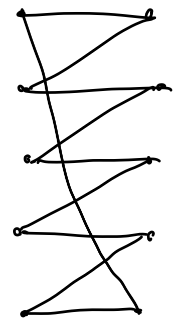
Def^n: An n-lacing is a closed path consisting of $2n$ line segments s.t.

- The endpoints of the segments are among A_i's, B_i's
- Exactly 2 segments should meet at each eyelet
- No 3 consecutive eyelets in the path lie in the same column.

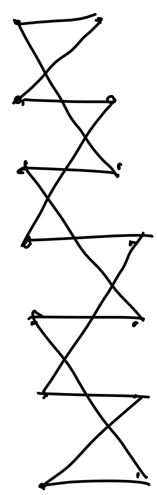
Examples:



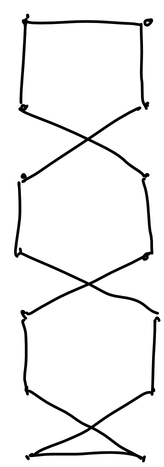
criss cross lacing



zig zag lacing



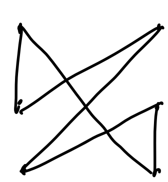
star lacing



bowtie lacing

Def^n:

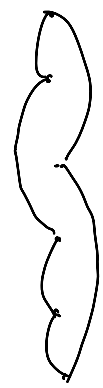
- A vertical is a segment both of whose endpoints are in the same column.
- A lacing is called simple if the 2 paths from A_1 to A_n involve vertices with non-decreasing indices.



· A dense lacing is one without verticals.

Counting lacings

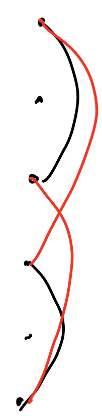
Toy example: One column lacings



Total number of lacings $= \frac{1}{2}(n-1)!$

Shortest lacings: Lacings of length $2(n-1)$
Number of such lacings $= 2^{n-3}$

Longest lacings:
Suppose n is even.



All eyelets in the top half are joined to eyelets in the bottom half.

Length of longest lacing $= \frac{n^2}{2}$

Number of such lacings $= \frac{1}{2} \left(\frac{n}{2}\right)! \left(\frac{n}{2}-1\right)!$

For odd n, length $= \frac{n^2-1}{2}$

Number of lacings $= \left(\left(\frac{n-1}{2}\right)!\right)^2$

<u>Counting lacings</u>

Number of dense lacings :

Give an example.

Every traversal looks like $ABABAB\ldots\ldots$

No. of choices $= \frac{1}{2} n! (n-1)!$

Number of general n-lacings :

<u>Lemma</u> : Number of verticals in column A equals numbers of verticals in column B.

Easy to prove.

Let $S(k,n)$ denote the no. of lacings with k verticals.

Fix a traversal of an n-lacing.

Then, it consists of blocks of A's and B's such that:

- There are $n-k$ A-blocks and $n-k$ B-blocks.
- For A (resp. B), there are k AA blocks and $n-k$ A blocks.

No. of traversals that start with the block "A"

$$= \binom{n-k-1}{k}\binom{n-k}{k}$$

" " " " " "AA"

$$= \binom{n-k-1}{k-1}\binom{n-k}{k}$$

No. of indices of the first one $= n!\,(n-1)!$

" " " " " second one $= 2 \cdot n!\,(n-1)!$

Hence, net number

$$= \frac{n!\,(n-1)!}{2} \cdot \sum_{k=0}^{(n-1)/2} \left[\binom{n-k-1}{k}\binom{n-k}{k} + 2\binom{n-k-1}{k-1}\binom{n-k}{k}\right]$$

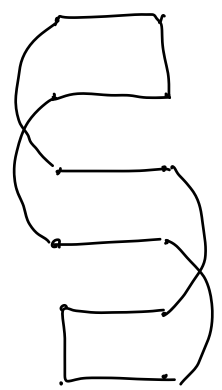
. . .

$$= \frac{(n!)^2}{2} \sum_{k=0}^{(n-1)/2} \frac{1}{n-k} \binom{n-k}{k}^2$$

Straight n-lacings: Contain all possible horizontal segments.

Superstraight n-lacings: Only contain horizontal and vertical segments.

Examples:

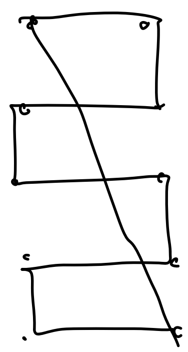


Lengths of Lacings

Theorem:

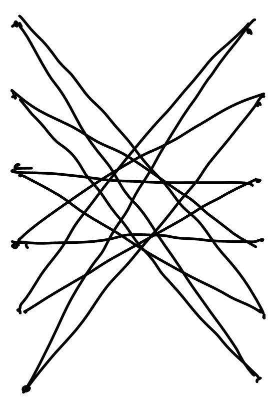
- Bowtie lacings are the shortest, in general.
- Crisscross lacings are the shortest dense lacings.
- All superstraight lacings are the shortest straight lacings for n-even.

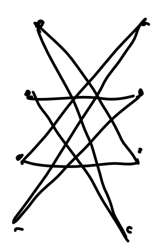
- Zigsag lacings are the shortest straight lacings for n - odd.



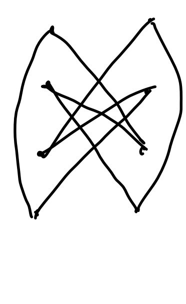
Theorem :

- For short shoes, devil lacings are the longest.

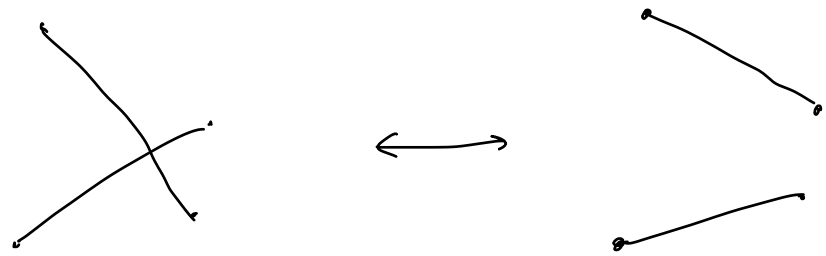




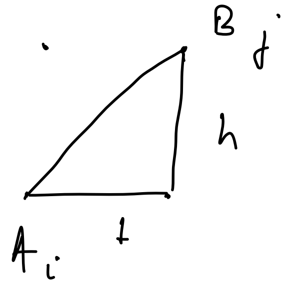
- For long shoes, it is conjectured that angel lacings are the longest.



$\exists\, h_n$ s.t. for $h < h_n$, devil lacings are the longest. For $h > h_n$, angel > devil, but may not be the longest.



Strongest lacings



Suppose the tension in the string is T.

Horizontal component $= \dfrac{T}{|A_iB_j|}$

We define the strength of an n-lacing l as

$$T_h(l) = \sum_{d \in \{\text{diagonals of } l\}} \frac{1}{|d|}$$

Theorem : There exists $h_n > 0$ s.t.

- For $h < h_n$, criss-cross lacing is the strongest.
- For $h > h_n$, zig-zag lacing is the strongest.

Remark : Outside their optimality zone, these crossings aren't even second best in general.

n	3	4	5	6	7	...
h_n	0.90	0.74	0.65	0.58	0.53	

Proof idea :

Let V be the collection of the vertical lengths of the diagonals in an n-lacing.

We want to maximise $$\sum_{s \in V} \frac{1}{\sqrt{1+(hs)^2}}$$

Rules :

- If V contains less than n elements, add n-m 0's and n-1 1's.
  If V contains $n \leq m < 2n$ elements, add $2n-m-1$ 1's and 1 n-1.

- If V contains elements whose sum exceeds $2(n-1)$, replace one v by $v-1$, and we improve the sum.

- Given $v, v' \in V$ s.t. $v + v' \geq 2$, if $\exists\, w, w'$ s.t.
  - $0 \leq w, w' \leq n-1$
  - $v + v' = w + w'$
  - $$\frac{1}{\sqrt{1+(hv)^2}} + \frac{1}{\sqrt{1+(hv')^2}} < \frac{1}{\sqrt{1+(hw)^2}} + \frac{1}{\sqrt{1+(hw')^2}},$$

  replace v, v' by w, w'.

<u>Lemma</u>: The set V determines the criss-cross and zig-zag lacings uniquely.

<u>Lemma</u>: For sets V other than the 2 above, it is always possible to perform one of the rules.

h_n satisfies
$$n-2+\frac{1}{\sqrt{1+(n-1)^2h^2}} - \frac{n-1}{\sqrt{1+h^2}} = 0.$$
