Kac's conjecture on absolutely indecomposable representations of quivers

Pallav Goyal

Discussed with Professor Victor Ginzburg

1 Introduction

The study of quivers is motivated by their connection to a wide range of problems in various fields of mathematics. Quivers and their representations provide for a uniform formulation of some important problems in linear algebra and representation theory. Every finite dimensional k-algebra is Morita equivalent to the path algebra kQ of some quiver Q modulo some relations, which potentially makes the study of representations easier because of the nice homological properties of quivers. Apart from this, Ringel discovered that quiver representations are related to Kac-Moody Lie algebras, which inspired the study of associated geometric objects called quiver varieties.

One of the first major results proven in the theory of quivers was Gabriel's striking classification of quivers of finite type, that is, quivers having finitely many indecomposable representations upto isomorphism (see [7]). Gabriel established that the underlying undirected graph of a quiver of finite type must be a Dynkin quiver, and that the indecomposable representations of such quivers are in one-to-one correspondence with the positive roots associated to the root data of the given Dynkin quiver. Kac widely generalized this result by providing a description for the dimensions of indecomposable representations of any arbitrary quiver (see [11, Section 3, Theorem 1′]).

In [12], Kac formulated the following two conjectures regarding absolutely indecomposable representations of a quiver over finite fields: Given a quiver Q and a dimension α, if $n_\alpha(q)$ denotes the number of absolutely indecomposable representations of Q over $\mathbb{F}_q$ having dimension α, then:

- $n_\alpha(q)$ is a polynomial in q with non-negative integer coefficients.
- The constant term of the above polynomial equals the multiplicity of α as a root in the Kac-Moody algebra associated to Q.

These conjectures were eventually proven for the case when α is indivisible in [4], and then, in general in [9] and [10].

Below, we'll give an outline of the proof of the first conjecture in the case when the dimension α is indivisible. The argument here almost entirely follows that given in [4] and is as follows. The idea will be to prove that $n_\alpha(q)$ is equal to the Poincare polynomial of a certain variety X_s (to be defined in Section 6.1) times a power of q. This will be done by showing that $n_\alpha(q)$ counts the $\mathbb{F}_q$-points on X_s (Proposition 6.4), and then, counting the points on X_s by using the Grothendieck-Lefschetz trace formula for the Frobenius action on l-adic cohomology with compact support (Theorem 7.2).

2 Definitions and setup

Most of the definitions in this section have been taken from [15]. A *quiver* is a 5-tuple (Q, E, V, h, t), where Q denotes a directed graph, E its set of edges, V its set of vertices and $h, t : E \to V$ respectively denote the head and tail functions. By abuse of notation, this quiver may simply be referred to as Q. A representation V of Q over a field k is a set of finite-dimensional k-vector spaces $(V_i)_{i\in V}$ and k-linear maps $\phi_e : V_{h(e)} \to V_{t(e)}$ for every $e \in E$. The sequence of dimensions $\alpha = (\alpha_i)_{i\in V} \in \mathbb{Z}^V$, where $\alpha_i = \dim_k(V_i)$, is called the dimension vector or, simply, the dimension of the representation V. The dimension is called *indivisible* if the GCD of the α_i's is 1. The *Tits form* q_Q associated to the quiver Q is a quadratic form on $\mathbb{Z}^V$ defined as: $(x_i)_{i\in V} \mapsto \sum_{i\in E} x_i^2 - \sum_{e\in E} x_{h(e)}x_{t(e)}$.

The k-representations of a quiver Q can also be seen as modules over a certain k-algebra kQ, known as the *path algebra* of Q. The set of oriented paths in Q forms a basis for the underlying vector space of kQ, whereas, multiplication of two paths is defined to be their concatenation, whenever it makes sense, and zero otherwise. Interpreting representations this way, we have the natural concepts of morphisms between representations, subrepresentations, direct sums and quotients of representations, irreducible (or simple) and indecomposable representations. We denote by $\operatorname{Rep}(Q,\alpha)$ the space of all representations of Q having dimension α.

A representation $x \in \operatorname{Rep}(Q,\alpha)(\mathbb{F}_q)$ is called *absolutely indecomposable* if $x \otimes k \in \operatorname{Rep}(Q,\alpha)(k)$ is indecomposable, where $k = \overline{\mathbb{F}_q}$. We denote by $\operatorname{Rep}(Q,\alpha)^{a.i.}$ the set of absolutely indecomposable representations. Define $n_\alpha(q)$ to be the number of elements in $\operatorname{Rep}(Q,\alpha)^{a.i.}(\mathbb{F}_q)$ upto isomorphism.

The space $\operatorname{Rep}(Q,\alpha)$ is a vector space isomorphic to $\prod_{e\in E} \operatorname{Hom}(k^{t(e)}, k^{h(e)})$ and has a natural linear action of the reductive group $\prod_{i\in V} GL_{\alpha_i}$ by conjugation. As the scalar matrices $(\lambda I_{\alpha_i})_{i\in V} \in \prod_{i\in V} GL_{\alpha_i}$ act trivially on the variety, we consider the action of the quotient group which we denote by $G(\alpha)$.

3 Deformed preprojective algebras

Given a quiver Q, we associate with it a double quiver $\overline{Q}$ which is obtained by taking the quiver Q and adding for each edge e in it, an edge e^* in the opposite direction. Then, $\operatorname{Rep}(\overline{Q},\alpha)$ too has the action of $G(\alpha)$ by conjugation. Given $\lambda \in k^V$, we define the corresponding deformed *preprojective algebra* as:

$$\Pi^\lambda := k\overline{Q}/\Big(\sum_{a\in E}[x_a, x_{a^*}] - \sum_{i\in V}\lambda_i I_{\alpha_i}\Big).$$

The Lie algebra of $G(\alpha)$ is a Lie subalgebra of $\prod_{i\in V} M_{\alpha_i\times\alpha_i}$ consisting of the space of trace zero matrices, and is denoted by $\mathfrak{g}$. We define the following map:

$$\mu : \operatorname{Rep}(\overline{Q},\alpha) \to \mathfrak{g} : (x_i)_{i\in\overline{Q}} \mapsto \sum_{a\in E}[x_a, x_{a^*}]$$

(where, by abuse of notation, we have identified $\overline{Q}$ with its set of edges). The vector space $\operatorname{Rep}(\overline{Q},\alpha)$ can be identified with the cotangent bundle of $\operatorname{Rep}(Q,\alpha)$, and thus has a natural symplectic structure. The $G(\alpha)$-action on $\operatorname{Rep}(\overline{Q},\alpha)$ turns out to be Hamiltonian and the above map can actually be interpreted as the *moment map* corresponding to this action.

If we have a non-zero Π^λ-representation of dimension α, then from the defining relation of Π^λ, we get $0 = \operatorname{Tr}(\sum_{i\in V}\lambda_i I_{\alpha_i}) = \lambda.\alpha$, where $(.,.)$ is the Euclidean inner product in $\mathbb{R}^V$. In fact, if we let λ denote the element $(\lambda_i I_{\alpha_i})_{i\in V} \in \mathfrak{g}$, we have that $\mu^{-1}(\lambda) = \operatorname{Rep}(\Pi^\lambda,\alpha)$.

4 Counting representations

Fix a quiver Q and an indivisible dimension $\alpha = (\alpha_i)_{i\in V}$. A vector $\lambda \in \mathbb{Z}^V$ is called *generic* with respect to α if $\lambda.\alpha = 0$ and $\lambda.\beta \neq 0$ for any β such that $\beta_i \leq \alpha_i$ for all i, with $\beta \neq 0, \alpha$. Such a λ can be chosen if and only if α is indivisible. Fix any such λ. Finally, fix a prime power q and let $k := \overline{\mathbb{F}_q}$. For indivisible dimensions, the notions of indecomposability and absolute indecomposability coincide, because if a representation splits in a finite extension of $\mathbb{F}_q$, all the subrepresentations will be Galois conjugates and, thus, have the same dimension.

In this section, we'll prove the following proposition:

Proposition 4.1. $n_\alpha(q) = q^{q_Q(\alpha)-1}|\operatorname{Rep}(\Pi^\lambda,\alpha)(\mathbb{F}_q)|/|G(\alpha)(\mathbb{F}_q)|$.

4.1 Relating Π^λ-representations to Q representations

The imbedding $kQ \hookrightarrow \Pi^\lambda$ induces a map on the representations in the opposite direction, which we denote by $\pi : \mathrm{Rep}(\Pi^\lambda, \alpha) \to \mathrm{Rep}(Q, \alpha)$. A representation x of Q is said to have a lift to Π^λ if there exists a Π^λ-representation y such that $\pi(y) = x$.

Lemma 4.2. *$x \in \mathrm{Rep}(Q, \alpha)$ has a lift if and only if it is an indecomposable representation.*

Proof. We consider the following short exact sequence associated with any representation $x = (x_e)_{e\in E} \in \mathrm{Rep}(Q, \alpha)$: (see [3, Lemma 3.1])

$$0 \to \mathrm{Ext}^1(x, x)^* \to \mathrm{Rep}(Q^{op}, \alpha) \xrightarrow{\phi} \mathrm{End}(\alpha) \xrightarrow{\theta} \mathrm{End}(x)^* \to 0. \tag{1}$$

Here, Q^{op} represents the quiver obtained by reversing the edges of Q, $\mathrm{End}(\alpha)$ denotes the space $\prod_{i\in V} M_{\alpha_i\times\alpha_i}$ and $\mathrm{End}(x)$ is the space of quiver endomorphisms of x. In this sequence, the maps θ and ϕ can be described explicitly as:

$$\phi : \mathrm{Rep}(Q^{op}, \alpha) \to \mathrm{End}(\alpha) : (y_{e^*})_{e\in E} \mapsto \sum_{e\in E} [x_e, y_{e^*}],$$

$$\theta : \mathrm{End}(\alpha) \to \mathrm{End}(x)^* : (\gamma_i)_{i\in V} \mapsto [(\delta_i)_{i\in V} \mapsto \sum_{i\in V} \mathrm{Tr}(\gamma_i\delta_i)].$$

From these, we get that x has a lift to Π^λ if and only if there exists a representation y of Q^{op} such that $\phi(y) = \lambda$, which happens if and only if for all endomorphisms $\delta = (\delta_i)_{i\in V}$ of x, we have $\theta(\lambda)(\delta) = \sum_{i\in V} \lambda_i \mathrm{Tr}(\delta_i) = 0$.

For any $y \in \mathrm{Rep}(\Pi^\lambda, \alpha)$, we want to show that $\pi(y)$ is an indecomposable representation of Q. Suppose $\pi(y) = x' \oplus x''$ is a decomposition as Q-representations. Then, the projection $\pi(y)$ onto x' is an endomorphism. By the previous paragraph, we must have $0 = \sum_{i\in V} \lambda_i \mathrm{Tr}(\delta_i) = \lambda.\beta$, where β denotes the dimension of x'. By genericity, we must have $\beta = \alpha$ or $\beta = 0$, proving that $\pi(y)$ is indecomposable.

Conversely, let x be an indecomposable representation of Q. We need to show that for any $\delta \in \mathrm{End}(x)$, we have $\theta(\lambda)(\delta) = 0$. Now, any such δ can be written as $\kappa + n$, where κ is an isomorphism and n is nilpotent. Due to absolute indecomposability, Fitting Lemma gives that κ must be of the form $c + n'$ for a scalar c and a nilpotent endomorphism n'. Then, we have $\theta(\lambda)(\delta) = \theta(\lambda)(c) = c\sum_{i\in V} \lambda_i\alpha_i = c(\lambda.\alpha) = 0$, thus proving the claim. ☐

If $x \in \mathrm{Rep}(Q, \alpha)$ has a lift, the sequence (1) implies that there are exactly $|\mathrm{Ext}^1(x, x)^*| = |\mathrm{Ext}^1(x, x)|$ lifts. Next, we prove the fact that $\dim(\mathrm{End}(x)) - \dim(\mathrm{Ext}^1(x, x))$ is independent of the choice of $x \in \mathrm{Rep}(Q, \alpha)^{a.i.}$ This is seen by taking the Euler characteristic of the exact sequence, to get that:

$$\begin{aligned}\dim(\mathrm{End}(x)) - \dim(\mathrm{Ext}^1(x, x)) &= \dim(\mathrm{End}(x)^*) - \dim(\mathrm{Ext}^1(x, x)^*)\\ &= \dim(\mathrm{End}(\alpha)) - \dim(\mathrm{Rep}(Q^{op}, \alpha))\\ &= \sum_{i\in V} \alpha_i^2 - \sum_{e\in E} \alpha_{h(e)}\alpha_{t(e)}\\ &= q_Q(\alpha),\end{aligned}$$

which is independent of the choice of x.

4.2 Translation to a geometric setting

Proof of Proposition 4.1. Two representations V and V' of Q having dimension α are isomorphic if and only if they lie in the same orbit under the $G(\alpha)$ action. Hence, the set of orbits $\mathrm{Rep}(Q, \alpha)/G(\alpha)$ can be identified with the isomorphism classes of representations of Q. By Burnside's counting lemma, we get:

$$|\mathrm{Rep}(Q, \alpha)^{a.i.}(\mathbb{F}_q)/G(\alpha)(\mathbb{F}_q)| = \frac{1}{|G(\alpha)(\mathbb{F}_q)|} \sum_{x\in \mathrm{Rep}(Q,\alpha)^{a.i.}(\mathbb{F}_q)} |\mathrm{Stab}_{G(\alpha)}(x)|.$$

Next, we claim that $q.|\mathrm{Stab}_{G(\alpha)}(x)| = |\operatorname{End}(x)|$ for all x. To prove this, we observe that every element in $\operatorname{End}(x)$ is of the form $c+n$ for some scalar c and some nilpotent endomorphism n (by Fitting Lemma). Of these, the invertible elements are those where $c \neq 0$. Thus, if the number of nilpotent endomorphisms in $\operatorname{End}(x)$ is t, we have $|\operatorname{End}(x)| = qt$, whereas $|G(\alpha)| = (q-1)t/(q-1) = t$ (as we quotient out the scalars), and so, $q.|\mathrm{Stab}_{G(\alpha)}(x)| = |\operatorname{End}(x)|$. Hence,

$$\begin{aligned}
|\operatorname{Rep}(Q,\alpha)^{a.i.}(\mathbb{F}_q)/G(\alpha)(\mathbb{F}_q)| &= \frac{1}{|G(\alpha)(\mathbb{F}_q)|} \sum_{x \in \operatorname{Rep}(Q,\alpha)^{a.i.}(\mathbb{F}_q)} \frac{1}{q} |\operatorname{End}(x)| \\
&= \frac{1}{q.|G(\alpha)(\mathbb{F}_q)|} \sum_{x \in \operatorname{Rep}(\Pi^\lambda,\alpha)(\mathbb{F}_q)} \frac{|\operatorname{End}(\pi(x))|}{|\operatorname{Ext}^1(\pi(x),\pi(x))|} \\
&= \frac{1}{q.|G(\alpha)(\mathbb{F}_q)|} \sum_{x \in \operatorname{Rep}(\Pi^\lambda,\alpha)(\mathbb{F}_q)} q^{q_Q(\alpha)} \\
&= q^{q_Q(\alpha)-1} \frac{|\operatorname{Rep}(\Pi^\lambda,\alpha)(\mathbb{F}_q)|}{|G(\alpha)(\mathbb{F}_q)|}.
\end{aligned}$$

□

5 Geometric Invariant Theory

The theory of this section has almost entirely been taken from [8]. Given the action of a reductive group G on an affine variety $X = \operatorname{Spec}(A)$, the naive quotient X/G doesn't, generally, behave nicely. One fix to this problem is to work with the *categorical quotient* $X//G := \operatorname{Spec}(A^G)$, where A^G denotes the subring of invariants in A under the action of G. Alternatively, we can also define the *GIT quotient* of X by G. For this, we first fix a character $\chi : G \to k^\times$ and consider the space of χ^n*-semi-invariant* polynomials defined as:

$$A^{\chi^n} := \{f \in A : g.f = \chi(g)^n f \text{ for all } g \in G\}.$$

This allows us to define the graded ring: $A_\chi := \oplus_{n\geq 0} A^{\chi^n}$, and we define the GIT quotient of X under the G-action with the stability condition χ to be $X//_\chi G := \operatorname{Proj}(A_\chi)$. When we take $\chi = 1$, this reduces to the categorical quotient defined above.

A point $x \in X$ is said to be χ*-semistable* if for some $n \geq 1$, there exists a χ^n-semi-invariant $f \in k[X]$ such that $f(x) \neq 0$. Such a point x is called *stable* if its G-orbit is closed and its stabiliser in G is finite. The set of χ-semistable in X is denoted by X^χ. Two points semistable points x and x' are said to be S-equivalent if and only if $\overline{G.x}$ and $\overline{G.x'}$ intersect in X^χ. Now, the inclusion $A_\chi \hookrightarrow A$ induces a map $\pi : X^\chi \to X//_\chi G$. Then, we have the following result (see [18, Theorem 3.14]):

Proposition 5.1. 1. *The map π induces a one-to-one correspondence between the S-equivalence classes of G-orbits in X and the geometric points of $X//_\chi G$.*

2. *The image under π of the set of stable points is a smooth subvariety of $X//_\chi G$. In particular, if every semistable point is stable, $X//_\chi G$ is smooth.*

The following characterisation of semistable points is due to Mumford, and popularly referred to as the Hilbert-Mumford criterion:

Theorem 5.2 (see [13], Theorem 1.4)**.** *Given a k-point x of X and G-stable subvariety S of X, then $G.x$ and S intersect if and only if there exists a one parameter subgroup θ of G, such that $\lim_{t\to 0}\theta(t).x$ exists and lies in S.*

In the context of quivers, we can consider stable and semistable representations in $\operatorname{Rep}(Q,\alpha)$ under the action of $G(\alpha)$. For any $\theta = (\theta_i)_{i\in V} \in \mathbb{Z}^V$, we can associate a character $\chi_\theta : G(\alpha) \to k^\times$ such that $\chi(g) = \prod_{i\in V} \det(g_i)^{\theta_i}$ for all $g = (g_i)_{i\in V} \in G(\alpha)$. In this situation, we have the followung consequence of the Hilbert-Mumford criterion:

Theorem 5.3 (King. [14], Proposition 3.1)**.** *A point $x \in \operatorname{Rep}(Q,\alpha)$ is χ_θ-semistable if and only if we have $\theta.\beta \geq 0$ for all $\beta \in \mathbb{Z}^V$ such that x has a proper subrepresentation having dimension β. The representation x is stable under the same conditions with strict inequality holding for every proper subrepresentation.*

6 Counting points on varieties

The following lemma, along with Proposition 4.1, implies that $n_\alpha(q) = q^{q_Q(\alpha)-1}|X(\mathbb{F}_q)|$, where $X := \operatorname{Rep}(\Pi^\lambda,\alpha)^\lambda//G(\alpha)$.

Lemma 6.1. $|X(\mathbb{F}_q)| = |\operatorname{Rep}(\Pi^\lambda,\alpha)(\mathbb{F}_q)|/|G(\alpha)(\mathbb{F}_q)|$

Proof. If we have $x \in \operatorname{Rep}(\Pi^\lambda,\alpha)(\mathbb{F}_q)$ and a subrepresentation y of x whose dimension is β, we must have $\lambda.\beta = 0$. Hence, for sufficiently large q, by genericity of λ, we get that x must be simple. Thus, x doesn't have any nilpotent endomorphisms proving that $\operatorname{End}(x) = \mathbb{F}_q$, and so, $\operatorname{Stab}_{G(\alpha)(k)}(x)$ is trivial. Thus, $|\operatorname{Rep}(\Pi^\lambda,\alpha)(\mathbb{F}_q)|/|G(\alpha)(\mathbb{F}_q)| = |\operatorname{Rep}(\Pi^\lambda,\alpha)(\mathbb{F}_q)/G(\alpha)(\mathbb{F}_q)|$.

Next, we consider the natural map:

$$\phi : \operatorname{Rep}(\Pi^\lambda,\alpha)(\mathbb{F}_q)/G(\alpha)(\mathbb{F}_q) \to \Big(\operatorname{Rep}(\Pi^\lambda,\alpha)(k)/G(\alpha)(k)\Big)^{Gal(k/\mathbb{F}_q)}.$$

We claim that this map is a bijection. Injectivity follows from Noether-Deuring theorem (see [17, Theorem 19.25]), which basically says that under the given conditions, representations which become isomorphic on the extension of scalars must have been isomorphic to start with. (The proof of this theorem is a linear algebraic argument that involves the Krull-Schmidt theorem.) For surjectivity, we note that for any orbit on the right hand side we have representations over k that are invariant (upto isomorphism) under the Galois action, and so by [16, Lemma 5.3.2], the field of definition of such representations must be $\mathbb{F}_q$, implying that they lie in the image of ϕ.

As each representation in $\operatorname{Rep}(\Pi^\lambda,\alpha)(k)$ is simple, by King's theorem we get that every representation is stable, and thus, all $G(\alpha)(k)$-orbits are closed. Now, the k-valued points of the categorical quotient $\operatorname{Rep}(\Pi^\lambda,\alpha)//G(\alpha)$ correspond to the closed orbits in $\operatorname{Rep}(\Pi^\lambda,\alpha)$ under the $G(\alpha)$-action under the usual categorical quotient map. So, we have the chain of equalities:

$$|\operatorname{Rep}(\Pi^\lambda,\alpha)(\mathbb{F}_q)/G(\alpha)(\mathbb{F}_q)| = |(\operatorname{Rep}(\Pi^\lambda,\alpha)(k)/G(\alpha)(k))^{Gal(k/\mathbb{F}_q)}| = |X(k)^{Gal(k/\mathbb{F}_q)}| = |X(\mathbb{F}_q)|,$$

completing the proof. □

6.1 Shifting to a different variety

In order to use the trace formula to count points on our variety, we will shift to a different variety that corresponds to representations of the undeformed preprojective algebra, unlike X. This will make it easier to count points later using the trace formula.

Consider the subvariety $X \subseteq \operatorname{Rep}(\overline{Q},\alpha) \times k$ whose ideal of definition is generated by polynomials corresponding to the equation $\mu(x) = s\lambda$ for $x \in \operatorname{Rep}(\overline{Q},\alpha)$ and $s \in k$. We get a $G(\alpha)$-action on $\operatorname{Rep}(\overline{Q},\alpha) \times k$ by making it act trivially on the second coordinate, and this gives a $G(\alpha)$-action on X. Let $Z := X^\lambda$.

Lemma 6.2. *Z is a smooth variety.*

Proof. Pick any $x \in \operatorname{Rep}(\overline{Q},\alpha)^\lambda$ with $\mu(x) = s\lambda$ for some $s \in k$. Let ϕ be any endomorphism of x. Then, if we have β and β' as the dimensions of $\ker(\phi)$ and $\operatorname{Im}(\phi)$ respectively, we get $\beta + \beta' = \alpha$, and so, genericity of λ and semistability imply that one of β or β' must be zero and the other must be α, which means that every endomorphism of x is either zero or a bijection. Then, by absolute indecomposability, we get that $\operatorname{End}(x) = k$, which means that the action of $G(\alpha)$ on x is free. Then, by [4, Lemma 2.1.5], we get that the moment map is smooth at x. Thus, Z is smooth. □

Due to genericity, every semistable representation is stable (see [4]), and so, we get that the GIT quotient $\mathcal{X} = Z//G(\alpha)$ is a smooth variety. In fact, if we consider the map $\pi : \mathcal{X} \to k$ induced by the projection of Z onto the second coordinate, we get that each fiber is a non-singular quasi-projective variety.

Next, take the G_m action on Z defined by $t.(x,s) = (tx, t^2 s)$. This induces an action of G_m on $\mathcal{X}$, and the map π is G_m-equivariant. Also, as $t \to 0$, every point in Z goes to $(0,0)$, which may not be an element of Z, but exists in the quotient $\mathcal{X}$, and so, the limit of $t.z$ as $t \to 0$ exists for all $z \in \mathcal{X}$. The reason for constructing such a G_m-action is the following result:

Proposition 6.3 (Nakajima)**.** *Suppose we are given a smooth variety $\mathcal{X}$ and a morphism $\pi : \mathcal{X} \to k$ such that every fibre is a non-singular quasi-projective variety. Suppose we have a G_m action on $\mathcal{X}$ that is G_m-equivariant, and for every $x \in \mathcal{X}$, the limit $\lim_{t\to 0} t.x$ exists. Then, for sufficiently large q, the number of $\mathbb{F}_q$ points in $\mathcal{X}$ above any $\gamma \in \mathbb{F}_q$ is independent of the choice of γ, provided that π isn't the constant zero map.*

Proof. For any $c \in k$, let $X_c = \pi^{-1}(c)$. Then, X_1 is isomorphic to X_t for all $t \in k^\times$, and so, we only need to compare $|X_1|$ and $|X_0|$. Let $\mathcal{X}^{G_m} = \sqcup_\alpha \mathcal{F}_\alpha$ be the decomposition into connected components. Then, we have the Bialynicki-Birula decompostion $\mathcal{X}$ into a disjoint union of components $\mathcal{X}_\alpha$, such that each $\mathcal{X}_\alpha$ is an affine fibration over $\mathcal{F}_\alpha$ (see [1, Theorem 4.1, Proof of Theorem 4.2]). Let the dimension of the fibration $\mathcal{X}_\alpha/\mathcal{F}_\alpha$ be n_α. Then,

$$|\mathcal{X}(\mathbb{F}_q)| = \sum_\alpha |\mathcal{X}_\alpha(\mathbb{F}_q)| = \sum_\alpha |\mathcal{F}_\alpha(\mathbb{F}_q)| q^{n_\alpha}.$$

As a point in $\mathcal{X}$ is fixed under the G_m-action only if it projects to 0 under π (due to the G_m-equivariance of π), we have $\mathcal{X}^{G_m} \subseteq \mathcal{X}_0$, and so, we have a similar decomposition $\mathcal{X}_0 = \sqcup_\alpha (\mathcal{X}_0)_\alpha$. Then, $(\mathcal{X}_0)_\alpha$ is again an affine fibration over $\mathcal{F}_\alpha$, this time of dimension $n_\alpha - 1$. Hence, $|\mathcal{X}_0(\mathbb{F}_q)| = \frac{1}{q}|\mathcal{X}(\mathbb{F}_q)| = \frac{1}{q}\sum_{t\in\mathbb{F}_q} |\mathcal{X}_t(\mathbb{F}_q)| = \frac{q-1}{q}|\mathcal{X}_1(\mathbb{F}_q)| + \frac{1}{q}|X_0(\mathbb{F}_q)|$, and thus, $|\mathcal{X}_0(\mathbb{F}_q)| = |\mathcal{X}_1(\mathbb{F}_q)|$. $\square$

Remark 6.1. The above proposition is not applicable when $\pi^{-1}(1)(\mathbb{F}_q)$ is empty. But, in that case, $|X(\mathbb{F}_q)| = 0$, which is a polynomial in q with non-negative coefficients, and so, the same is true for $n_\alpha(q)$.

Applying this proposition to our situation, we have that $|X(\mathbb{F}_q)| = |\pi^{-1}(1)(\mathbb{F}_q)| = |\pi^{-1}(0)(\mathbb{F}_q)| = |X_s(\mathbb{F}_q)|$, where $X_s := \mathrm{Rep}(\Pi^0, \alpha)^\lambda//G(\alpha)$, for sufficiently large q. Thus, we have:

Proposition 6.4. $n_\alpha(q) = q^{q_Q(\alpha)-1}|X_s(\mathbb{F}_q)|$.

6.2 Purity

We want to define the Frobenius morphism for a scheme $\pi : Z \to \mathrm{Spec}(k)$. First, define the map $G : Z \to Z$ which is the identity map on topological spaces, whereas on the structure sheaf, $G^\# : \mathcal{O}(Z) \to \mathcal{O}(Z)$ is the q^{th}-power map. Then, G is a morphism of schemes, but isn't k-linear. Define the scheme Z' to be the same as Z, but with the structural morphism equal to $\pi \circ G = G' \circ \pi$, where G' is the Frobenius morphism for $\mathrm{Spec}(k)$. Then the map G becomes a k-linear morphism $F : Z' \to Z$, and this k-morphism F is called the Frobenius morhpism on Z (after identifying the schemes Z and Z').

Now, given a variety Z and a prime l, we can define the étale cohomology groups $H^i(Z, \mathbb{Z}/l^k\mathbb{Z})$ for all $k \geq 1$ (see [6, Chapter 1]). Then, the l-adic cohomology groups of Z are defined as:

$$H^i(Z, \mathbb{Q}_l) := \varprojlim_k H^i(Z, \mathbb{Z}/l^k\mathbb{Z}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

Finally, the l-adic cohomology of Z with compact support is constructed by taking an open immersion $j : Z \to X$ for some proper scheme X, and defining $H^i_c(Z, \mathbb{Q}_l) := H^i(X, j_!\mathbb{Q}_l)$, where $j_!\mathbb{Q}_l$ denotes extension by zero of the sheaf $\mathbb{Q}_l$. This definition is independent of the choice of the immersion j. If Z is a variety over k of dimension n, the Frobenius morphism on Z induces an action F^* on the l-adic cohomology with compact support (for any l coprime to q), and we say that Z is pure if all the eigenvalues of F^* due to its action on $H^i_c(Z, \mathbb{Q}_l)$ have absolute value $q^{i/2}$, for all $0 \leq i \leq 2n$.

Weil conjectures ([5, Corollary 3.3.9]) imply that a smooth, projective variety over k is pure, but this won't be sufficient for us as X_s isn't necessarily projective. In order to show that Z is pure, we'll be using the following proposition:

Proposition 6.5. *Suppose Z is a smooth, quasi-projective variety with a G_m action, such that for all $x \in Z$, the limits $\lim_{t\to 0} t.x$ exists in Z. Also, suppose the fixed point set Z^{G_m} is a projective variety. Then, Z is cohomologically pure.*

Proof. Let $Z^{G_m} = \sqcup_\alpha L_\alpha$ be the decompositon into connected components. Then, we have the Bialynicki Birula decomposition $Z = \sqcup_\alpha Z_\alpha$, such that Z_α and L_α are smooth, each Z_α is an affine fibration over L_α, and we have a filtration $\emptyset = Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \ldots \subseteq Z_n = Z$ where each Z_i is a closed subset of Z such that for each i, $Z_{i+1} \setminus Z_i$ is Z_α for some α (see [1, Theorem 4.1, Proof of Theorem 4.2] and [2, Proof of Theorem 3]).

L_α is pure because of Weil conjectures. As W_α is an affine fibration over L_α, their l-adic cohomology is isomorphic, and so, by Poincare duality, the eigenvalues of the action of the Frobenius on l-adic cohomology with compact support are the same in the two cases, implying that W_α is pure. Then, gluing together the Z_α's according to the above filtration, we get that Z is pure. $\square$

In order to apply this proposition to our variety X_s, we consider the action of G_m on $\mathrm{Rep}(\Pi^0, \alpha)^\lambda$ by defining $t.x$ for any $t \in G_m$ and $x \in \mathrm{Rep}(\Pi^0, \alpha)^\lambda$ to be the representation whose linear maps are t times those of x. Then, consider the action induced by this on X_s. This is a restriction of the action of G_m defined on $\mathcal{X}$ in Section 6.1. Hence, the purity of X_s follows from the following proposition:

Proposition 6.6. *In the above setting and notation, $X_s^{G_m}$ is a projective variety.*

Proof. For this, consider the variety $X_s^0 = \mathrm{Rep}(\Pi^0, \alpha)//G(\alpha)$. Then, the map $\phi : X_s \to X_s^0$, induced by the natural map $\mathrm{Rep}(\Pi^0, \alpha)^\lambda \to \mathrm{Rep}(\Pi^0, \alpha)$, is a projective morphism. Also, our action of G_m on X_s extends to an action on X_s^0, making the above map G_m-equivariant. Now, $X_s^0 = \mathrm{Spec}(A^{G(\alpha)})$, where A is the ring of regular functions on $\mathrm{Rep}(\Pi^0, \alpha)$. $A^{G(\alpha)}$ is graded under the action of G_m, and each homogeneous part is a finite dimensional k-vector space. Now, the only prime ideal that remains fixed under the action of G_m is the maximal ideal $(A^{G(\alpha)})_+$, and so, $(X_s^0)^{G_m}$ consists of a single point and is, thus, projective. Therefore, $(X_s)^{G_m} = \phi^{-1}((X_s^0)^{G_m})$ is projective, completing the proof. Therefore, $X_s = \mathrm{Rep}(\Pi^0, \alpha)^\lambda//G(\alpha)$ is pure. $\square$

7 Grothendieck-Lefschetz trace formula

We need the following lemma:

Lemma 7.1. *Given complex numbers $z_1, z_2, \ldots, z_n$ such that $|z_i| = 1$ for all i, suppose the limit $\lim_{r\to\infty} \sum_{i=1}^n z_i^r$ exists. Then, $z_i = 1$ for all i.*

Proof. We proceed by induction on n. Let L be the above limit. If $L = 0$, we can show that $\lim_{r\to\infty} \sum_{i=1}^n (z_i/z_1)^r = 0$. As the first term in the sum is 1, we are done by induction. Henceforth, suppose $L \neq 0$. Consider the polynomial $p(t) = \prod_{i=1}^n (t - z_i) = \sum_{i=0}^n c_i t^i$ for some $c_i \in \mathbb{C}$. Then, for all r, $0 = \sum_{j=1}^n z_j^r p(z_r) = \sum_{j=1}^r \sum_{i=0}^n c_i z_j^{i+r}$. Taking the limit of this expression as $r \to \infty$, we get $0 = L \sum_{i=0}^n c_i = Lp(1)$, and as $L \neq 0$, we get $p(1) = 0$ showing that $z_i = 1$ for some i. Thus, by induction, the claim stands proven. $\square$

From Proposition 6.4, we have that $n_\alpha(q) = q^{q_Q(\alpha)-1}|X_s(\mathbb{F}_q)|$. By [12, Proposition 1.15], we know that $n_\alpha(q)$ is a polynomial in q with integer coefficients. (The proof uses the Krull-Schmidt theorem and a recurrence relation between the number of absolutely indecomposable represetations of different dimensions.) Thus, we have that $|X_s(\mathbb{F}_q)| = \sum_{i\in S} b_i q^i$ for some $b_i \in \mathbb{Z}$ and S is a finite set of integers (possibly negative). We want to show that $b_i \geq 0$ for all i. The *Grothendieck-Lefschetz trace formula* (see [6, Theorem 3.1]) for smooth k-schemes of dimension d says that:

$$|X(\mathbb{F}_{q^r})| = \sum_{i=0}^{2d} (-1)^i \operatorname{Tr}(F^r; H_c^i(X_s, \mathbb{Q}_l)),$$

where $F : X_s \to X_s$ is the Frobenius and l is a prime not dividing q. We'll apply this formula for our variety X_s. Let $\beta_j = \dim_{\mathbb{Q}_l}(H_c^i(X_s, \mathbb{Q}_l))$. By purity, we know that all the eigenvalues of the action of F on $H_c^i(X_s, \mathbb{Q}_l)$ are of the form $\epsilon_{i,j} q^{i/2}$ for some $|\epsilon_{i,j}| = 1$ for all $1 \le j \le \beta_i$. Thus, we get:

$$\sum_{i \in S} b_i q^{ri} = |X_s(\mathbb{F}_q^r)| = \sum_{i=0}^{2\dim(X_s)} \sum_{j=1}^{\beta_i} \epsilon_{i,j}^r q^{ri/2}.$$

As both the sides are equal for all positive integers r, dividing by q^{rd} on both sides gives that $b_i = 0$ for all $i > d$, and $b_d = \lim_{r\to\infty} \sum_{j=1}^{\beta_{2d}} \epsilon_{2d,j}^r$. By the above lemma, we get that $\epsilon_{2d,j} = 1$ for all $1 \le j \le \beta_{2d}$. Subtracting the leading term corresponding to q^{rd} on both sides and proceeding in exactly the same fashion, we get that $\epsilon_{i,j} = 1$ for all i and j, and so, in particular, $b_i \ge 0$ for all i. This shows that all the coefficients of the polynomial n_α are non-negative, thus completing the proof.

Theorem 7.2. *For any prime power q, we have:*

$$n_\alpha(q) = q^{q_Q(\alpha)-1} \sum_{i=0}^{2d} \dim(H_c^{2i}(X_s, \mathbb{Q}_l)) q^i. \tag{2}$$

8 Examples

1. Consider the Dynkin quiver A_{n+1} with vertices $\{1, 2, \ldots, n+1\}$ and arrows $i \to i+1$ for $1 \le i \le n$. Let $\alpha = (1, 1, \ldots, 1)$. A linear map between one dimensional vector spaces is multiplication by a scalar, and so, a representation for Q of dimension α is the same as a tuple $(x_1, x_2, \ldots, x_n) \in \mathbb{F}_q^n$, where x_i corresponds to the map for the arrow $i \to i+1$. Such a representation of Q is indecomposable if and only if none of the linear maps $x_i \neq 0$. Upto isomorphism, there is just a unique such representation $(1, 1, \ldots, 1)$, and so, $n_\alpha(q) = 1$.

   Going by our strategy above, choose $\lambda = (1, 1, \ldots, 1, -n)$ which is generic with respect to α. Then, $n_\alpha(q) = q^{q_Q(\alpha)-1}|X(\mathbb{F}_q)|$, where $X = \mathrm{Rep}(\Pi^\lambda, \alpha)^\lambda // G(\alpha)$. We have $q_Q(\alpha) = n - (n-1) = 1$. Next, let P denote the ring of regular functions on $\mathrm{Rep}(\Pi^\lambda, \alpha)$. Then, P is a polynomial ring on $2n$ variables, one corresponding to each edge in the double quiver $\overline{Q}$, modulo the defining relation for Π^λ. More precisely:

   $$P = \frac{k[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n]}{(x_1y_1 - 1, x_2y_2 - x_1y_1 - 1, \ldots, x_ny_n - x_{n-1}y_{n-1} - 1, x_ny_n - n)}.$$

   As all the arrows correspond to invertible maps (they are multiplications by non-zero scalars), every representation of Π^λ is simple and, hence, stable. Therefore $X = \mathrm{Rep}(\Pi^\lambda, \alpha)//G(\alpha)$, and so,

   $$X = \mathrm{Spec}(P^{G(\alpha)}) = \mathrm{Spec}\left(\frac{k[x_1y_1, x_2y_2, \ldots, x_ny_n]}{(x_1y_1 - 1, x_2y_2 - x_1y_1 - 1, \ldots, x_ny_n - x_{n-1}y_{n-1} - 1, x_ny_n - n)}\right)$$

   which is isomorphic to $\mathrm{Spec}(k)$, and thus, $|X(\mathbb{F}_q)| = 1$, verifying that $n_\alpha(q) = 1$.

2. Consider the quiver with 2 vertices a and b, 1 loop at a and 1 edge from a to b. Let $\alpha = (1, 1)$. A representation for Q is indecomposable if and only if the linear map from a to b is non-zero. An isomorphism class of such indecomposable representations is chosen by normalising the edge from a to b to correspond to the linear map 1 and choosing the other map arbitrarily. It follows that $n_\alpha(q) = q$.

   Next, choose $\lambda = (1, -1)$. We have $q_Q(\alpha) = 1 - 1 = 0$. Proceeding as in the previous case, if A denotes the ring of regular functions on $\mathrm{Rep}(\Pi^\lambda, \alpha)$, we have:

   $$A = \frac{k[e, f, x, y]}{(ef - 1)},$$

where e and f correspond to the arrows between a and b, whereas x and y correspond to the two loops at a. Again, all representations are stable, and so,

$$X = \mathrm{Spec}(A^{G(\alpha)}) = \mathrm{Spec}\left(\frac{k[x, y, ef]}{(ef-1)}\right) \cong \mathrm{Spec}(k[x, y]) = \mathbb{A}^2,$$

and thus, $|X(\mathbb{F}_q)| = q^2$, giving that $n_\alpha(q) = q^{q_Q(\alpha)-1}|X(\mathbb{F}_q)| = q$.

Remark 8.1. In both of the examples, the final point count can also be verified from the trace formula using the fact that for the affine space $\mathbb{A}^n$, the cohomology is concentrated in dimension $2n$.

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